Positivity among *P*-partition enumerators

Peter McNamara
Bucknell University

Joint work with:
Nathan Lesnevich
Washington University in St. Louis

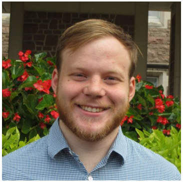
AMS Special Session on *Combinatorics and Computing*
4 October 2020

Slides and paper available from
`http://www.unix.bucknell.edu/~pm040/`

Positivity among *P*-partition enumerators

Peter McNamara
Bucknell University

Joint work with:
Nathan Lesnevich
Washington University in St. Louis



AMS Special Session on *Combinatorics and Computing*
4 October 2020

Slides and paper available from
`http://www.unix.bucknell.edu/~pm040/`

Outline

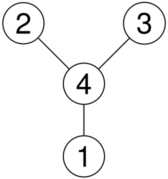
- Posets and the (P, ω)-partition enumerator
- Quasisymmetric functions and our main goal
- Summary of results

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P,ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P,\omega) =$

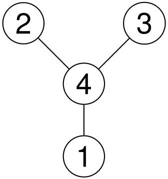


Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$



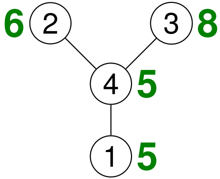
Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$ 

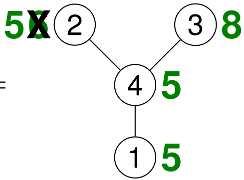
Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$ 

Not a (P, ω)-partition

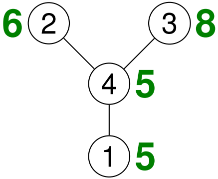
Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$ 

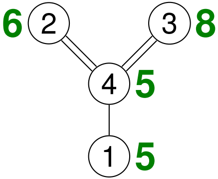
Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$ 

Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

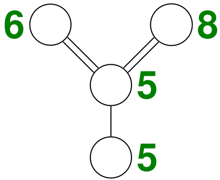
- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

We use double edges to denote the strictness conditions

Labeled posets

Poset: **p**artially **o**rdered **set**

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

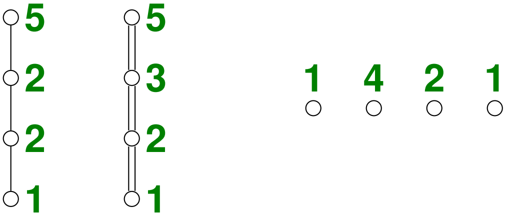
$(P, \omega) =$ 

Key definition. A (P, ω)-partition is a map f from P to the positive integers satsifying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

We use double edges to denote the strictness conditions and then we can (usually) ignore the underlying labeling.

Motivating examples for (P,ω)-partitions

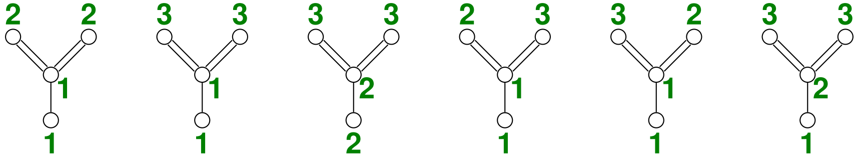


- (P,ω) chain with all weak edges: get a partition
- (P,ω) chain with all strict edges: get a partition with distinct parts
- (P,ω) is an antichain: get a composition

General (P,ω)-partitions interpolate between these classical objects.

The (P, ω)-partition enumerator

Example. Resrict to $f(p) \in \{1, 2, 3\}$.



$x_1^2 x_2^2$ $\quad$ $x_1^2 x_3^2$ $\quad$ $x_2^2 x_3^2$ $\quad$ $x_1^2 x_2 x_3$ $\quad$ $x_1^2 x_2 x_3$ $\quad$ $x_1 x_2 x_3^2$

$$K_{(P,\omega)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + 2x_1^2 x_2 x_3 + x_1 x_2 x_3^2.$$

In general, the (P, ω)-partition enumerator is by given by:

$$K_{(P,\omega)}(x_1, x_2, \ldots) = \sum_{(P,\omega)\text{-partition } f} x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots .$$

Equality question

$$K_{(P,\omega)}(x_1, x_2, \ldots) = \sum_{(P,\omega)\text{-partition } f} x_1^{\# f^{-1}(1)} x_2^{\# f^{-1}(2)} \cdots .$$

Equality question

$$K_{(P,\omega)}(x_1, x_2, \ldots) = \sum_{(P,\omega)\text{-partition } f} x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots .$$

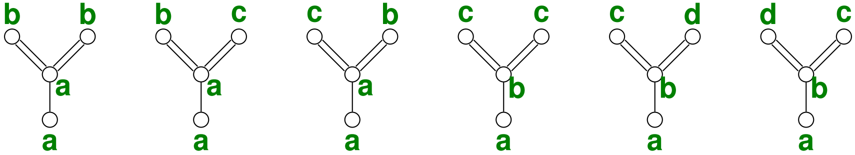
Open question. Determine simple necessary and sufficient conditions on labeled posets (P,ω) and (Q,τ) so that $K_{(P,\omega)} = K_{(Q,\tau)}$.

[Thomas Browning, Valentin Féray, Takahiro Hasebe, Max Hopkins, Zander Kelly, Ricky Liu, M., Shuhei Tsujie, Ryan Ward, Michael Weselcouch]

Generalizes the question of determining when two skew Schur functions are equal.

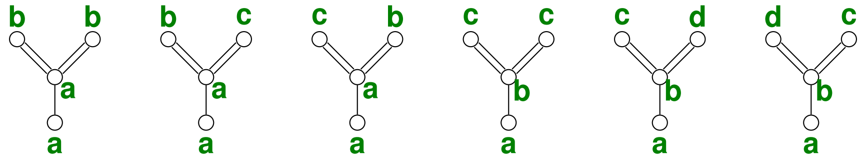
To state our goal, we need a little quasisymmetric background....

Same Example. But now with $f(p) \in 1, 2, \ldots$. With $a < b < c < d$, every (P, ω)-partition falls into one of these classes:



Quasisymmetric functions

Same Example. But now with $f(p) \in 1, 2, \ldots$. With $a < b < c < d$, every (P, ω)-partition falls into one of these classes:



$$\sum_{a<b} x_a^2 x_b^2 \qquad 2\sum_{a<b<c} x_a^2 x_b x_c \qquad \sum_{a<b<c} x_a x_b x_c^2 \qquad 2\sum_{a<b<c<d} x_a x_b x_c x_d$$

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ the monomial quasisymmetric function is:

$$M_\alpha = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

In our example, $K_{(P,\omega)} = M_{22} + 2M_{211} + M_{112} + 2M_{1111}$.

Quasisymmetric functions

The M_α form a basis for the quasisymmetric functions, stars of 21st century algebraic combinatorics.

Quasisymmetric functions

The M_α form a basis for the quasisymmetric functions, stars of 21st century algebraic combinatorics.

A more important basis for us is Gessel's fundamental quasisymmetric functions:

$$F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta .$$

e.g.

$F_{32} = M_{32} + M_{212} + M_{122} + M_{1112} + M_{311} + M_{2111} + M_{1211} + M_{11111}$.

(M_{221}, for example, does not appear).

Quasisymmetric functions

The M_α form a basis for the quasisymmetric functions, stars of 21st century algebraic combinatorics.
A more important basis for us is Gessel's fundamental quasisymmetric functions:

$$F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta.$$

e.g.
$F_{32} = M_{32} + M_{212} + M_{122} + M_{1112} + M_{311} + M_{2111} + M_{1211} + M_{11111}$.
(M_{221}, for example, does not appear).

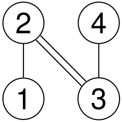
Why we care about F-basis:

1. One of the original two bases for quasisymmetric functions
2. Important symmetric function bases expand positively in F-basis
3. One of the candidates for a quasisymmetric analogue of Schur functions
4. Most importantly for us: Stanley & Gessel's $K_{(P,\omega)}$ expansion

Gessel & Stanley's expansion

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

Example.

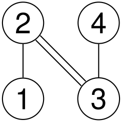


Linear extensions: $\mathcal{L}(P,\omega) = \{3412, 1324, 1342, 3124, 3142\}$.

Gessel & Stanley's expansion

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

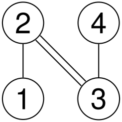
Example.



Linear extensions: $\mathcal{L}(P,\omega) = \{34|12, 13|24, 134|2, 3|124, 3|14|2\}$.

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

Example.



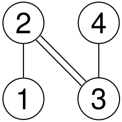
Linear extensions: $\mathcal{L}(P,\omega) = \{34|12, 13|24, 134|2, 3|124, 3|14|2\}$.

Descent compositions: comp(π) 22 22 31 13 121

Gessel & Stanley's expansion

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

Example.



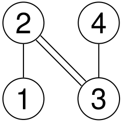
Linear extensions: $\mathcal{L}(P,\omega) = \{34|12, 13|24, 134|2, 3|124, 3|14|2\}$.
Descent compositions: comp(π) 22 22 31 13 121

$$K_{(P,\omega)} = 2F_{22} + F_{31} + F_{13} + F_{121}.$$

Gessel & Stanley's expansion

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

Example.



Linear extensions: $\mathcal{L}(P,\omega) = \{34|12, 13|24, 134|2, 3|124, 3|14|2\}$.
Descent compositions: comp(π) 22 22 31 13 121

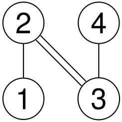
$$K_{(P,\omega)} = 2F_{22} + F_{31} + F_{13} + F_{121}.$$

Theorem [Gessel & Stanley] .For a labeled poset (P,ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

Stanley ('71) and Gessel ('84): $K_{(P,\omega)}$ expands beautifully in F-basis.

Example.



Linear extensions: $\mathcal{L}(P,\omega) = \{34|12, 13|24, 134|2, 3|124, 3|14|2\}$.
Descent compositions: comp(π) 22 22 31 13 121

$$K_{(P,\omega)} = 2F_{22} + F_{31} + F_{13} + F_{121}.$$

Theorem [Gessel & Stanley] .For a labeled poset (P,ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

Brigtwell & Winkler ('91): Counting linear extensions is #P-complete.

Our goal

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

So (P, ω)-parition enumerators are F-positive.

Our goal

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

So (P, ω)-parition enumerators are F-positive.

Question. When does one labeled poset have a "more F-positive" (P, ω)-parition enumerator than another.

Our goal

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So (P, ω)-parition enumerators are *F*-positive.

Question. When does one labeled poset have a "more *F*-positive" (P, ω)-parition enumerator than another.

Our goal. Determine simple necessary and sufficient conditions on labeled posets (P, ω) and (Q, τ) so that $K_{(Q,\tau)} - K_{(P,\omega)}$ is *F*-positive.

Our goal

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So (P, ω)-parition enumerators are F-positive.

Question. When does one labeled poset have a "more F-positive" (P, ω)-parition enumerator than another.

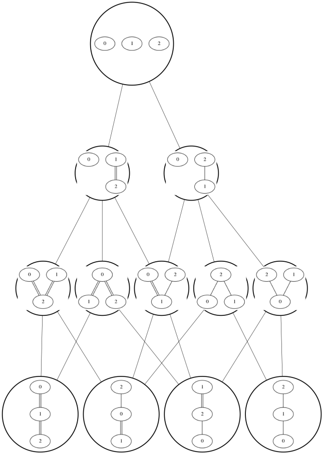
Our goal. Determine simple necessary and sufficient conditions on labeled posets (P, ω) and (Q, τ) so that $K_{(Q,\tau)} - K_{(P,\omega)}$ is F-positive.

Motivation.

- Positivity questions have (always?) been at the forefront of algebraic combinatorics
- Natural next question after the equality question
- Symmetric analogue has received a lot of attention ($\geq$ 15 papers)
- Representation theoretic: F-positive functions are characteristics of 0-Hecke algebra actions.

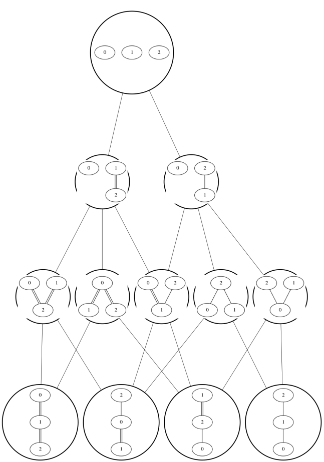
The F-positivty poset

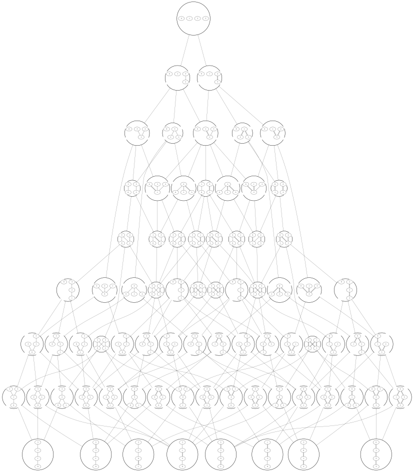
An ordering on labeled posets:
$(P, \omega) \leq_F (Q, \tau)$ if $K_{(Q,\tau)} - K_{(P,\omega)}$ is F-positive.



The F-positivty poset

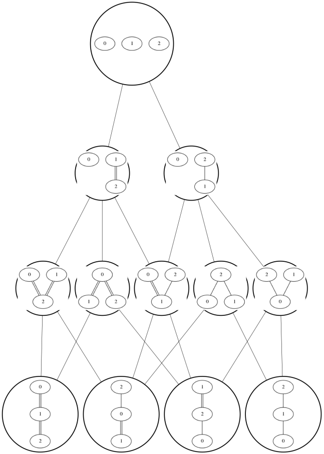
An ordering on labeled posets:
$(P,\omega) \leq_F (Q,\tau)$ if $K_{(Q,\tau)} - K_{(P,\omega)}$ is F-positive.

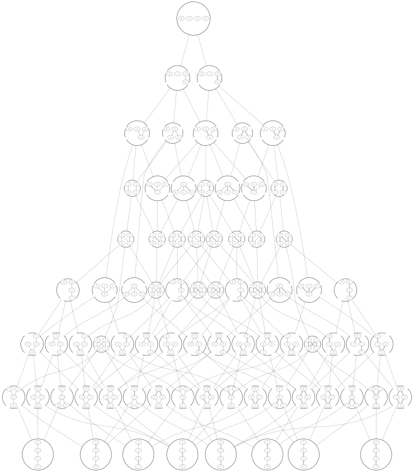




The F-positivty poset

An ordering on labeled posets:
$(P, \omega) \leq_F (Q, \tau)$ if $K_{(Q,\tau)} - K_{(P,\omega)}$ is F-positive.





Our goal restated. Understand these posets.

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

- Must have the same number of elements (degree of $K_{(P,\omega)}$).

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

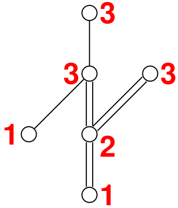
- Must have the same number of elements (degree of $K_{(P,\omega)}$).
- If (P, ω) has all weak (resp. strict) edges, so does (Q, τ).

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

- Must have the same number of elements (degree of $K_{(P,\omega)}$).
- If (P, ω) has all weak (resp. strict) edges, so does (Q, τ).
- The jump sequence of (Q, τ) must dominate that of (P, ω).



Jump sequence: $(2, 1, 3)$

Usual dominance order on compositions:

α dominates β if $\displaystyle\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for all k.

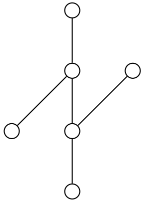
e.g., $(2, 1, 3)$ dominates $(1, 2, 2, 1)$

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

- Must have the same number of elements (degree of $K_{(P,\omega)}$).
- If (P, ω) has all weak (resp. strict) edges, so does (Q, τ).
- The jump sequence of (Q, τ) must dominate that of (P, ω).
- Suppose P and Q are natually labeled (all weak edges). Then the Greene shape of P dominates that of Q.



Greene shape: $(4, 2)$

Usual dominance order on compositions:

α dominates β if $\displaystyle\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for all k.

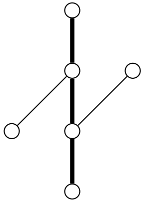
e.g., $(2, 1, 3)$ dominates $(1, 2, 2, 1)$

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

- Must have the same number of elements (degree of $K_{(P,\omega)}$).
- If (P, ω) has all weak (resp. strict) edges, so does (Q, τ).
- The jump sequence of (Q, τ) must dominate that of (P, ω).
- Suppose P and Q are natually labeled (all weak edges). Then the Greene shape of P dominates that of Q.



Greene shape: $(4, 2)$

Usual dominance order on compositions:

α dominates β if $\displaystyle\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for all k.

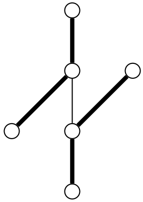
e.g., $(2, 1, 3)$ dominates $(1, 2, 2, 1)$

Necessary conditions

Since both the equality question and the symmetric analogue are still wide open, we aim for meaningful partial results.

Necessary conditions. If $(P, \omega) \leq_F (Q, \tau)$, what has to be true about (P, ω) versus (Q, τ)?

- Must have the same number of elements (degree of $K_{(P,\omega)}$).
- If (P, ω) has all weak (resp. strict) edges, so does (Q, τ).
- The jump sequence of (Q, τ) must dominate that of (P, ω).
- Suppose P and Q are natually labeled (all weak edges). Then the Greene shape of P dominates that of Q.



Greene shape: $(4, 2)$

Usual dominance order on compositions:

α dominates β if $\displaystyle\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$ for all k.

e.g., $(2, 1, 3)$ dominates $(1, 2, 2, 1)$

A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

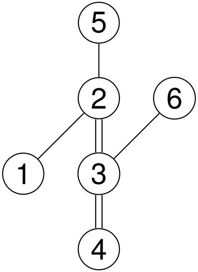
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ if and only if (Q, τ) can be obtained from (P, ω) by redundancy-before-deletion.



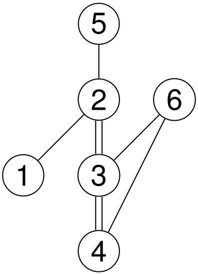
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ if and only if (Q, τ) can be obtained from (P, ω) by redundancy-before-deletion.



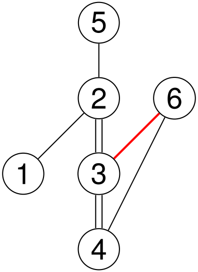
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

So, if $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$, then certainly $(P,\omega) \leq_F (Q,\tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$ if and only if (Q,τ) can be obtained from (P,ω) by redundancy-before-deletion.



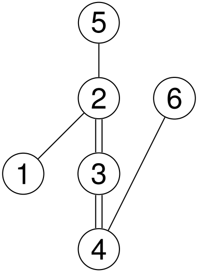
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}.$$

So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ if and only if (Q, τ) can be obtained from (P, ω) by redundancy-before-deletion.



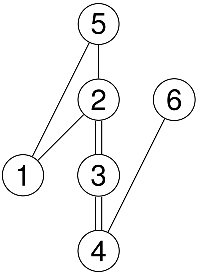
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So, if $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$, then certainly $(P,\omega) \leq_F (Q,\tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$ if and only if (Q,τ) can be obtained from (P,ω) by redundancy-before-deletion.



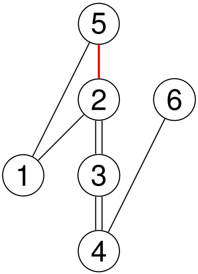
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ if and only if (Q, τ) can be obtained from (P, ω) by redundancy-before-deletion.



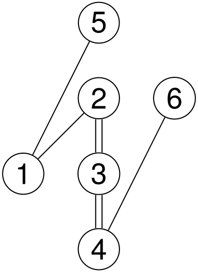
A sufficient condition

Theorem [Gessel & Stanley]. For a labeled poset (P, ω),

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\text{comp}(\pi)}.$$

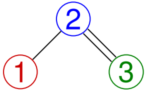
So, if $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$, then certainly $(P, \omega) \leq_F (Q, \tau)$.

Theorem [Lesnevich, M.]. $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ if and only if (Q, τ) can be obtained from (P, ω) by redundancy-before-deletion.

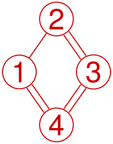


What operations on posets preserve F-positivity?

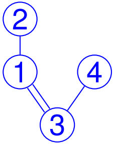
Poset assembly (called “Ur-operation” in [Browning, Hopkins, Kelly])



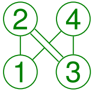
$(\mathcal{P},\omega)$



(P_1,ω_1)

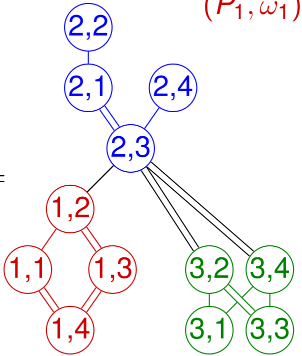


(P_2,ω_2)



(P_3,ω_3)

$(\mathcal{P}[i \to P_i]) =$



Poset assembly and F-positivity

Theorem [Lesnevich, M.].

- Labeled posets $(\mathcal{P}, \omega)$ and $(\mathcal{Q}, \tau)$ such that $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$.
- Sequences $((P_1, \omega_1), \ldots, (P_{|\mathcal{P}|}, \omega_{|\mathcal{P}|}))$ and $((Q_1, \tau_1), \ldots, (Q_{|\mathcal{P}|}, \tau_{|\mathcal{P}|}))$ of labeled posets satisfying $(P_r, \omega_r) \leq_F (Q_r, \tau_r)$ for all r.

Then

$$(\mathcal{P}[i \to P_i]) \leq_F (\mathcal{Q}[i \to Q_i]).$$

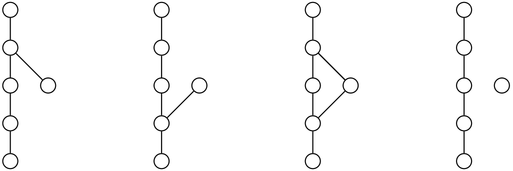
Notes.

- Computationally difficult to test.
- False if replace $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$ by $(\mathcal{P}, \omega) \leq_F (\mathcal{Q}, \tau)$. Counterexamples have 4×3 elements.
- $\mathcal{P} = \mathcal{Q} =$ 2-element antichain: disjoint union presevers F-positivity
- $\mathcal{P} = \mathcal{Q} =$ 2-element chain (with either strict or weak edge): ordinal sum preserves F-positivity

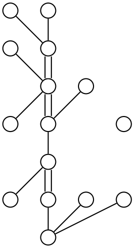
Special families

For 2 families of labeled posets, we have a full classification of $\leq_F$.

- Posets of Greene shape $(k, 1)$ with all weak edges:



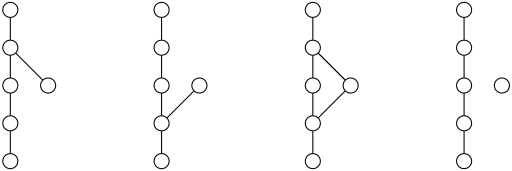
- Mixed-spine caterpillar posets:



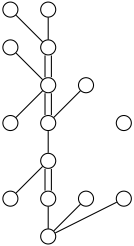
Special families

For 2 families of labeled posets, we have a full classification of $\leq_F$.

- Posets of Greene shape $(k, 1)$ with all weak edges:



- Mixed-spine caterpillar posets:



Thanks for your attention!

Closing remarks

- Families on previous slide are very special and there seems to be plenty of scope for stronger or related results.
- For simplicity of presentation, we've only talked about F-positivity, but many of results hold for M-positivity and/or F-support containment.
- In fact, a weakness of our necessary conditions is that they don't use the full power of F-positivity, and M-support containment is often enough.
- In the equality case, get stronger results by restricting to naturally labeled posets (all weak edges). We have only scratched the surface of the potential of this restriction for positivity.

Closing remarks

- Families on previous slide are very special and there seems to be plenty of scope for stronger or related results.
- For simplicity of presentation, we've only talked about F-positivity, but many of results hold for M-positivity and/or F-support containment.
- In fact, a weakness of our necessary conditions is that they don't use the full power of F-positivity, and M-support containment is often enough.
- In the equality case, get stronger results by restricting to naturally labeled posets (all weak edges). We have only scratched the surface of the potential of this restriction for positivity.

Thanks for your attention!