Two New Characterizations of Lattice Supersolvability

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Slides and paper available from

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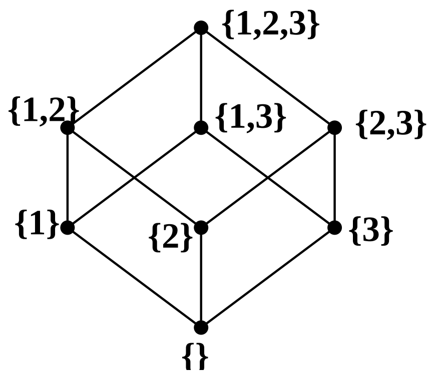
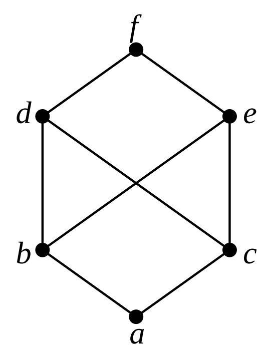
Definition A partially ordered set P is said to be a *lattice* if every two elements x and y of P have a least upper bound and a greatest lower bound.

We call the least upper bound the *join* of x and y and denote it by $x \vee y$.

We call the greatest lower bound the *meet* of x and y and denote it by $x \wedge y$.



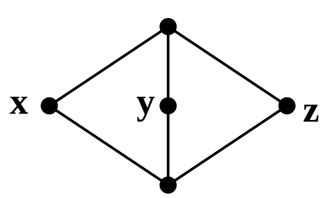


We say that a lattice L is *distributive* if

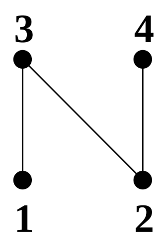
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \text{and}$$

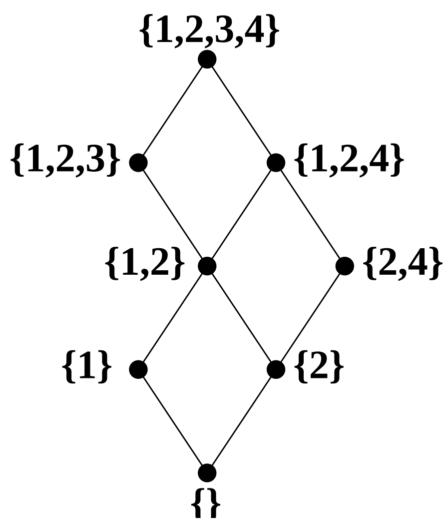
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all elements x, y and z of L.



Example An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The lattice of order ideals of a poset P is a distributive lattice.





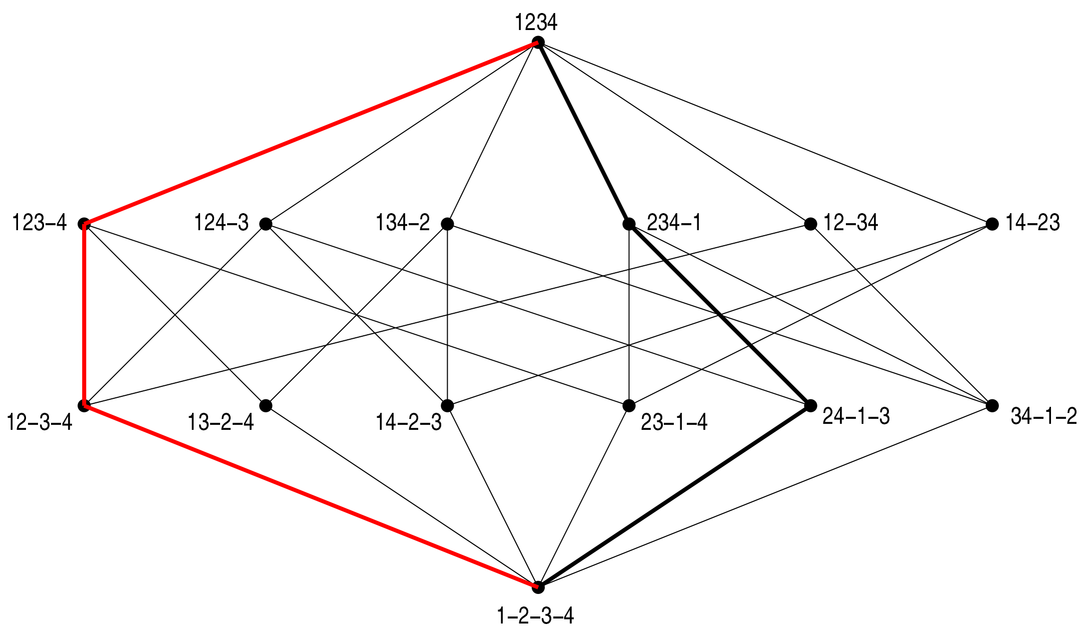
The Fundamental Theorem of Finite Distributive Lattices (Birkhoff):

Theorem *A finite lattice L is distributive if and only if it is the lattice of order ideals of some poset P.*

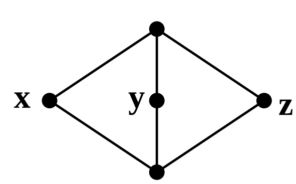
We write $L = J(P)$.

Definition (R. Stanley, '72) A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}$, called an *M-chain* of L which together with any other chain of L generates a distributive sublattice.

Examples



- Distributive lattices



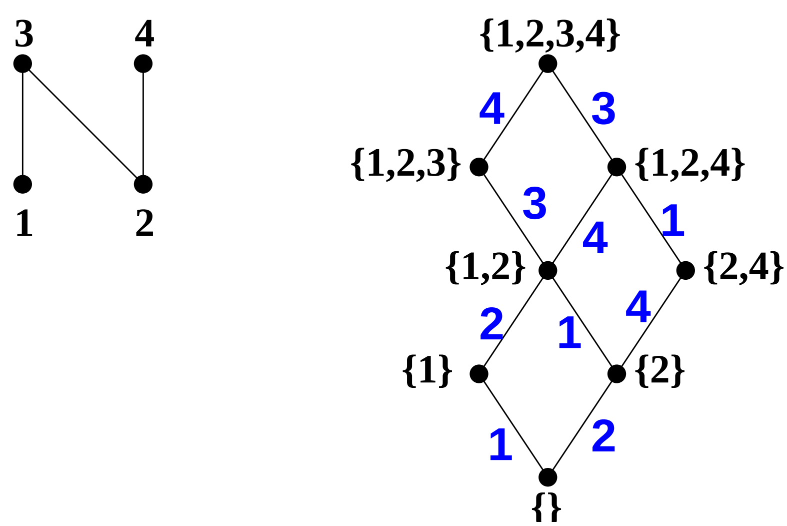
- The lattice $L(G)$ of subgroups of a supersolvable group G.

An edge-labeling of a poset P is said to be an *EL-labeling* if it satisfies the following 2 conditions:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels

2. This chain has the lexicographically least set of labels

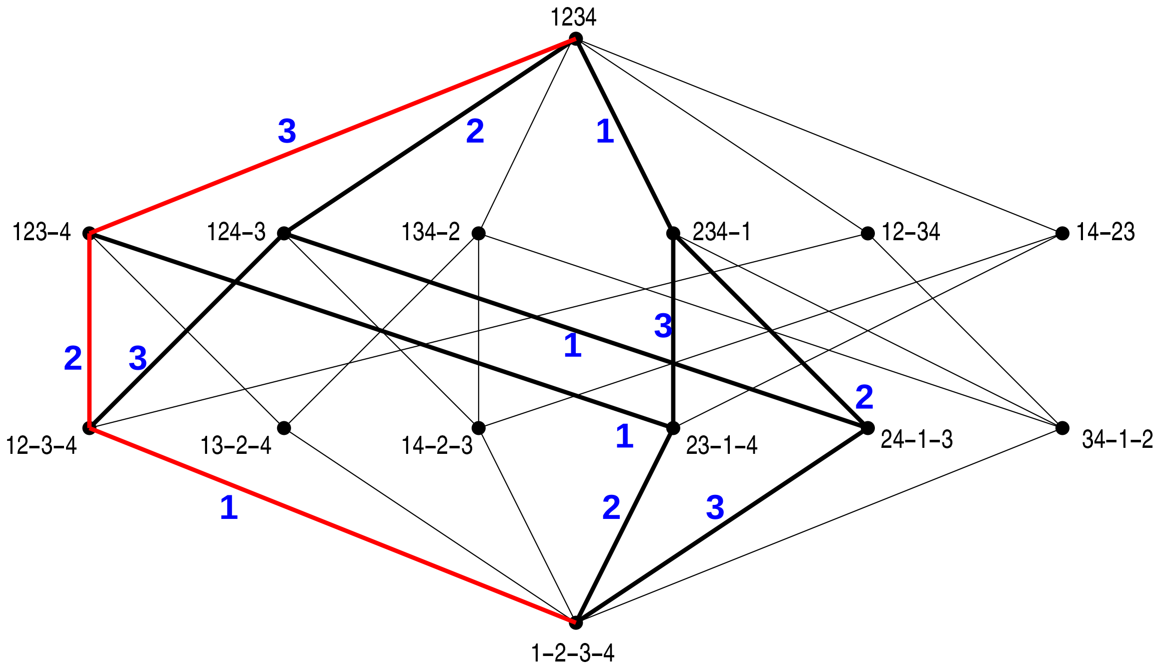
EXAMPLE



Why do we care?

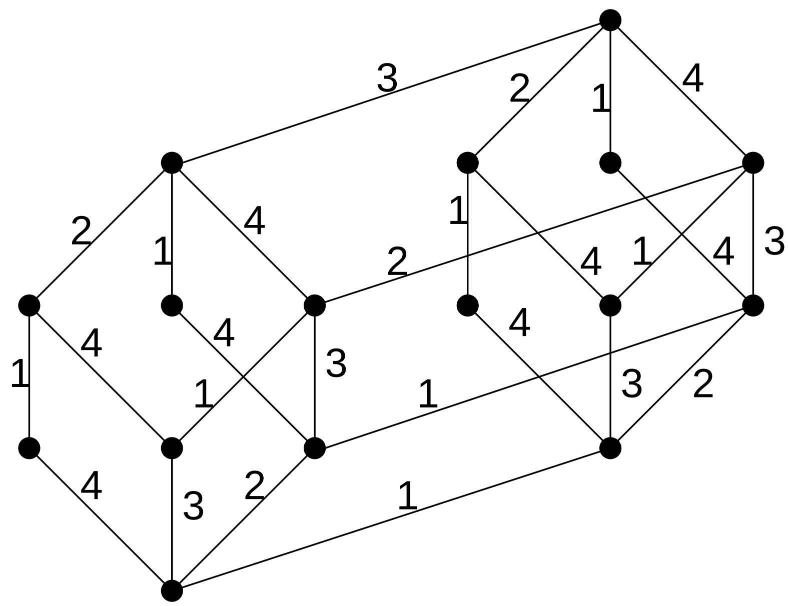
EL-labeling $\Rightarrow$ Shellable $\Rightarrow$ Cohen-Macaulay

Key Example Supersolvable Lattices



Remark Our EL-labelings of supersolvable lattices have the additional nice property that the labels along any maximal chain give a permutation of $[n]$.

In this case, we call our labeling an *S_n EL-labeling* or *snelling*, for short. If L has a snelling, then we say it is *S_n EL-shellable* or *snellable*, for short.



Stanley: "Could it be that L is supersolvable if and only if L has an S_n EL-labeling?"

THEOREM *A lattice is supersolvable if and only if it has an S_n EL-labeling.*

We want the chain $\mathfrak{m}_0$ with labels $1, 2, 3, \ldots, n$ to be an M-chain. Let $\mathfrak{m}$ be any other chain of L. (It suffices to consider only maximal chains.) The proof relies on the equivalence of the following 3 posets:

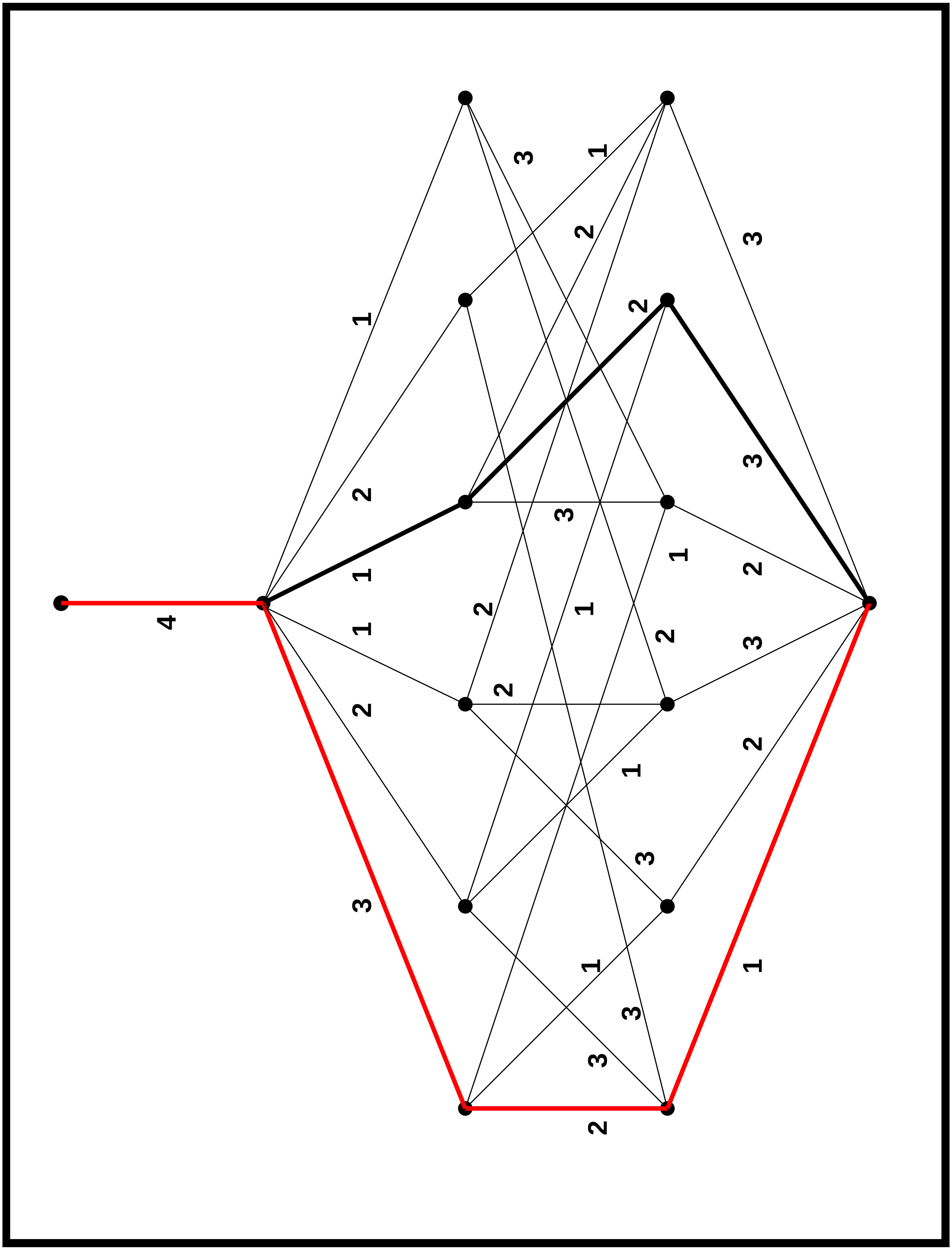
1. The sublattice $L_{\mathfrak{m}}$ of L generated by $\mathfrak{m}$ and $\mathfrak{m}_0$

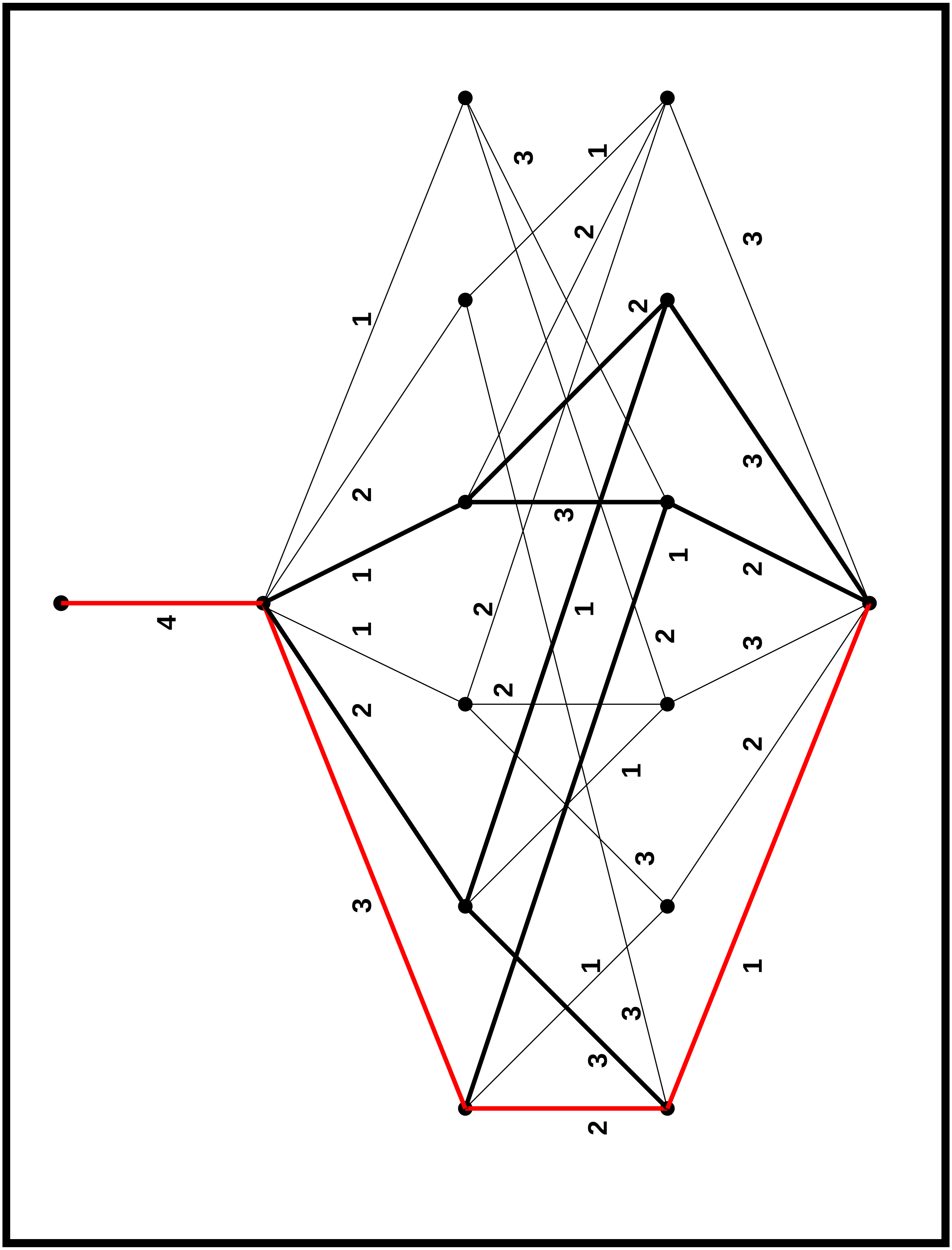
2. Let $\omega_{\mathfrak{m}}$ be the permutation labeling $\mathfrak{m}$. We construct a poset $P_{\omega_{\mathfrak{m}}}$ on the numbers $1, 2, \ldots, n$ defined by:

   $$i < j \text{ in } P_{\omega_{\mathfrak{m}}} \iff (i, j) \text{ isn't an inversion in } \omega_{\mathfrak{m}}$$

   for all $i < j$. Then we construct and label $J(P_{\omega_{\mathfrak{m}}})$ as before.

3. If $\mathfrak{m}$ has a descent at i, then we define $S_i(\mathfrak{m})$ to be the unique chain in L differing from $\mathfrak{m}$ only at rank i and having no descent at i. If $\mathfrak{m}$ doesn't have a descent at i then we set $S_i(\mathfrak{m}) = \mathfrak{m}$. We define $Q_{\mathfrak{m}}$ to be the "closure" of $\mathfrak{m}$ in L under the action of $S_1, S_2, \ldots, S_{n-1}$.





Leaving supersolvability behind...

Let P denote a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$ and with an S_n EL-labeling.

The action of $S_1, S_2, \ldots, S_{n-1}$ has the following properties:

1. It is a local action: $S_i(\mathfrak{m})$ equals $\mathfrak{m}$ except possibly at rank i
2. ${S_i}^2 = S_i$
3. $S_i S_j = S_j S_i$ if $|i - j| \geq 2$
4. $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$
5. $\mathrm{ch}(\chi_P(x)) = \omega(F_P(x))$

An action on the maximal chains of a lattice having all of these properties is called a *good $\mathcal{H}_n(0)$ action*.

"Good": Simion and Stanley.

What the Hecke does $\mathrm{ch}(\chi_P(x)) = \omega(F_P(x))$ mean?

P is a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$. Let $S \subseteq [n-1]$.

We let $\alpha_P(S)$ denote the number of chains in P whose elements, other than $\hat{0}$ and $\hat{1}$, have rank set equal to S.

$\alpha_P : 2^{[n-1]} \to \mathbb{Z}$ is called the *flag f-vector* of P

Define the *flag h-vector* β_P by

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T) \quad \text{or}$$

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

We define *Ehrenborg's flag function* by

$$F_P(x) = \sum_{\hat{0}=t_0 \leq \cdots \leq t_{k-1} < t_k = \hat{1}} x_1^{\mathrm{rk}(t_0,t_1)} \cdots x_k^{\mathrm{rk}(t_{k-1},t_k)}.$$

In general, it's a *quasisymmetric function*, i.e., for every sequence $n_1, n_2, \ldots, n_m$ of exponents, $x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m}$ and $x_{j_1}^{n_1} x_{j_2}^{n_2} \cdots x_{j_m}^{n_m}$ appear with equal coefficients whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$.

Fundamental quasisymmetric functions, $L_{S,n}(x)$:

$$L_{S,n}(x) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n} \ .$$

In this basis:

$$F_P(x) = \sum_{S \subseteq [n-1]} \beta_P(S) L_{S,n}(x) \ .$$

The involution ω for quasisymmetric functions:

$$\omega(L_{S,n}) = L_{[n-1]-S\ ,n}.$$

Then $\omega(s_\lambda) = s_{\lambda^t}$.

Background for $\text{ch}(\chi_P(x))$.....

Definition The *0-Hecke algebra $\mathcal{H}_n(0)$ of type A_{n-1}* is the $\mathbb{C}$-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ with relations:

(i) ${T_i}^2 = -T_i$ for $i = 1, 2, \ldots, n-1$.

(ii) $T_i T_j = T_j T_i$ if $|i - j| \geq 2$.

(iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $i = 1, 2, \ldots, n-2$.

Duchamp, Hivert, Krob, Leclerc, Thibon.

Setting, $S_i = -T_i$, we see that our action is a local $\mathcal{H}_n(0)$ algebra action.

- 2^{n-1} irreducible representations of $\mathcal{H}_n(0)$.
- All have dimension 1.
- They're labeled by subsets S of $[n-1]$.

Since $T_i^2 = -T_i$,

$$\psi_S(T_i) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

$$\psi_S(S_i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

The character χ_S of ψ_S:

$$\chi_S(S_{i_1} S_{i_2} \cdots S_{i_k}) = \begin{cases} 1 & \text{if } i_j \in S \text{ for } j = 1, \ldots, k \ , \\ 0 & \text{otherwise.} \end{cases}$$

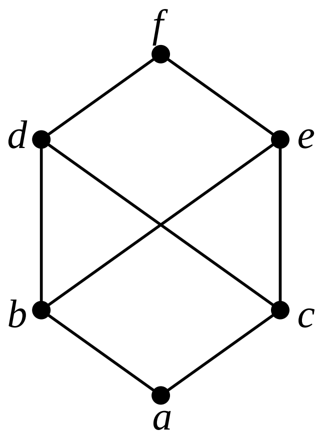
We let χ_P denote the character of the defining representation of our local $\mathcal{H}_n(0)$ action on $\mathbb{C}\mathcal{M}(P)$, the vector space over $\mathbb{C}$ with basis consisting of the maximal chains of P.

Following Krob and Thibon, we define its *characteristic* by $\mathrm{ch}(\chi_S) = L_{S,n}(x)$.

In the case when P has an S_n EL-labeling, $\mathrm{ch}(\chi_P(x)) = \omega(F_P(x))$ boils down to:
For all $S \subseteq [n-1]$, the number of maximal chains of P with "descent set" S equals $\beta_P(S)$.
[EC1, Thm. 3.13.2]

What other posets have good $\mathcal{H}_n(0)$ actions?

EXAMPLE



$\mathfrak{m}$	$S_1(\mathfrak{m})$	$S_2(\mathfrak{m})$
$\mathfrak{m}_1 : a < b < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_2$
$\mathfrak{m}_2 : a < b < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_2$
$\mathfrak{m}_3 : a < c < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_4$
$\mathfrak{m}_4 : a < c < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_4$

This gives a local $\mathcal{H}_n(0)$ action.

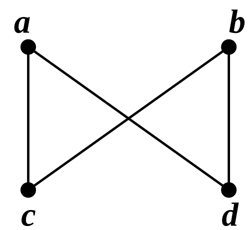
S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\alpha_P(S)$	1	2	2	4
$\beta_P(S)$	1	1	1	1

	1	S_1	S_2
$\chi_\emptyset$	1	0	0
$\chi_{\{1\}}$	1	1	0
$\chi_{\{2\}}$	1	0	1
$\chi_{\{1,2\}}$	1	1	1
χ_P	4	2	2

We see that $\chi_P = \chi_\emptyset + \chi_{\{1\}} + \chi_{\{2\}} + \chi_{\{1,2\}}$.
Therefore,

$$\begin{aligned} \mathrm{ch}(\chi_P) &= L_{\emptyset,3} + L_{\{1\},3} + L_{\{2\},3} + L_{\{1,2\},3} \\ &= F_P(x) \\ &= \omega F_P(x) \end{aligned}$$

Definition A graded poset P is said to be *bowtie-free* if it does not contain distinct elements a, b, c, d such that a covers both c and d, and such that b covers both c and d.



THEOREM *Let P be a finite graded bowtie-free poset of rank n with $\hat{0}$ and $\hat{1}$. Then P is S_n EL-shellable if and only if P has a good $\mathcal{H}_n(0)$ action.*

COROLLARY *Let L be a finite lattice. TFAE:*

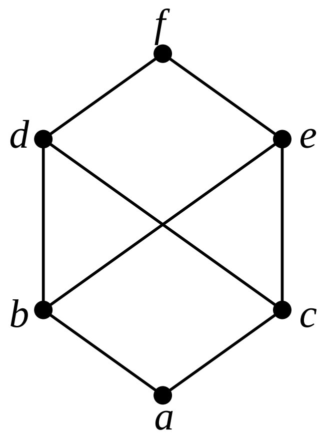
1. *L is supersolvable*
2. *L has an S_n EL-labeling*
3. *L has a good $\mathcal{H}_n(0)$ action*

Idea of proof (Stanley):

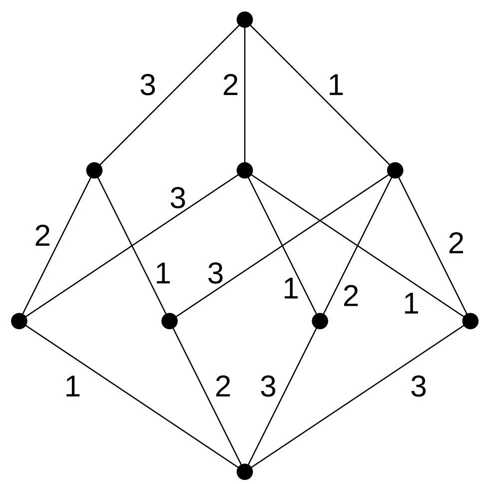
1. Suppose P has a unique chain $\mathfrak{m}_0$ fixed under $S_1, S_2, \ldots, S_{n-1}$.
2. Given $\mathfrak{m}$ we can find $S_{i_1}, S_{i_2}, \ldots, S_{i_r}$ with r minimal such that $S_{i_1} S_{i_2} \cdots S_{i_r}(\mathfrak{m}) = \mathfrak{m}_0$.
3. Define $\omega_{\mathfrak{m}} = s_{i_1} s_{i_2} \cdots s_{i_r}$. Then $\omega_{\mathfrak{m}}$ is well-defined.
4. Label the edges of m from bottom to top by $\omega_{\mathfrak{m}}(1), \omega_{\mathfrak{m}}(2), \ldots, \omega_{\mathfrak{m}}(n)$. This gives an edge-labeling of P and this edge-labeling is an S_n EL-labeling.

What about posets that aren't bowtie-free?

Example



Example



Question *Let $\mathcal{C}$ denote the class of finite graded posets with $\hat{0}$, $\hat{1}$ and a good $\mathcal{H}_n(0)$ action. Is there some "nice" characterization of $\mathcal{C}$, possibly in terms of edge-labelings?*