Tilings from the floor up

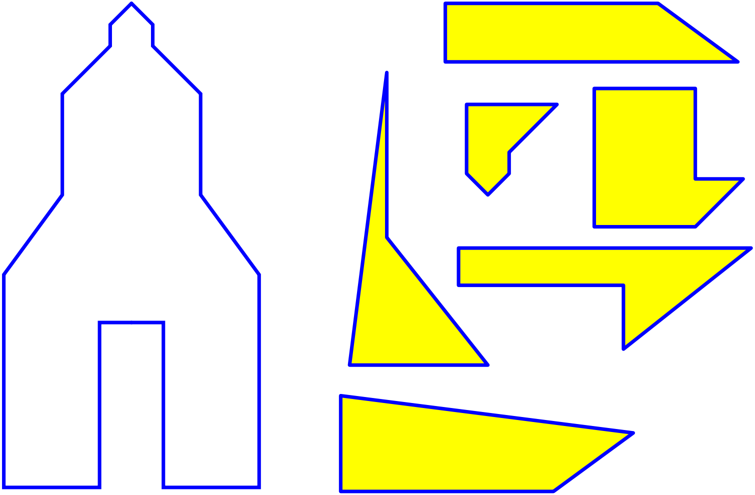
Peter McNamara
Bucknell University

Dublin University Mathematical Society
6 February 2013

Slides available (soon) from
www.facstaff.bucknell.edu/pm040/

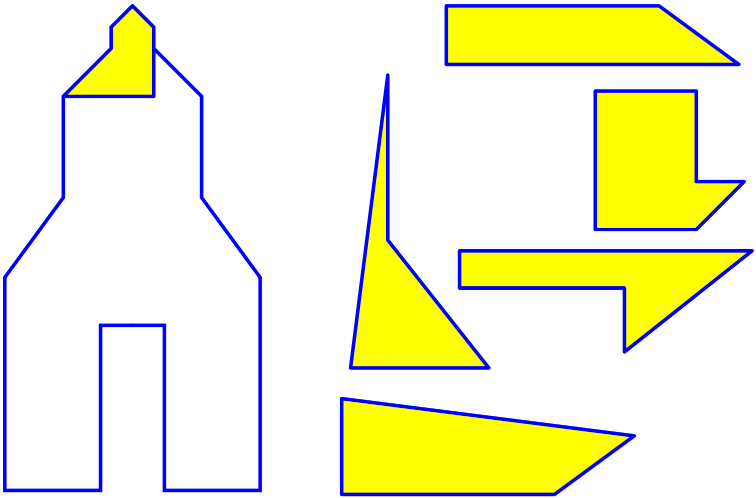
What is a tiling?

Tangrams:



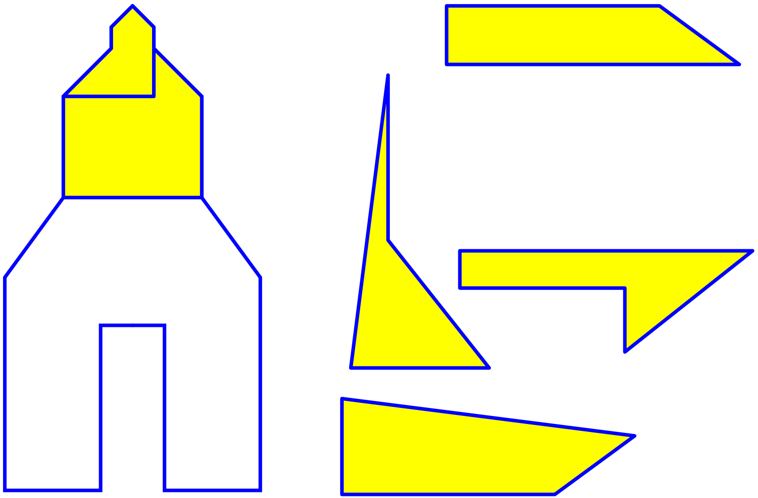
What is a tiling?

Tangrams:



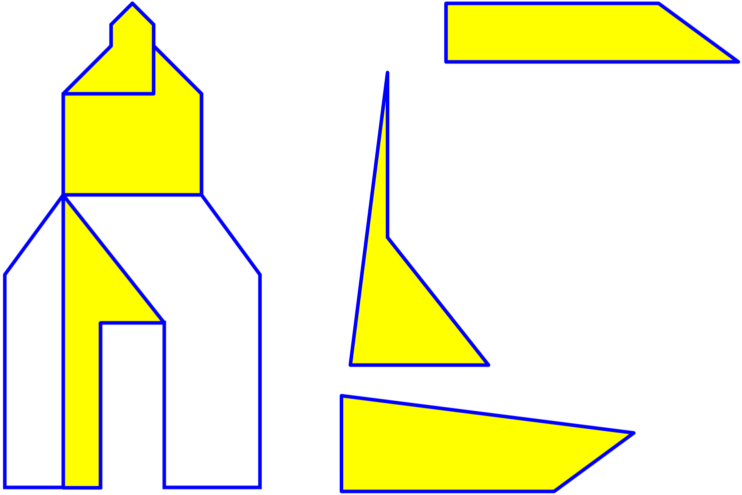
What is a tiling?

Tangrams:



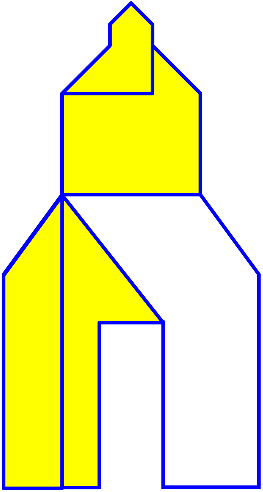
What is a tiling?

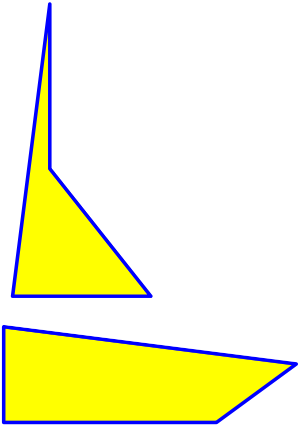
Tangrams:



What is a tiling?

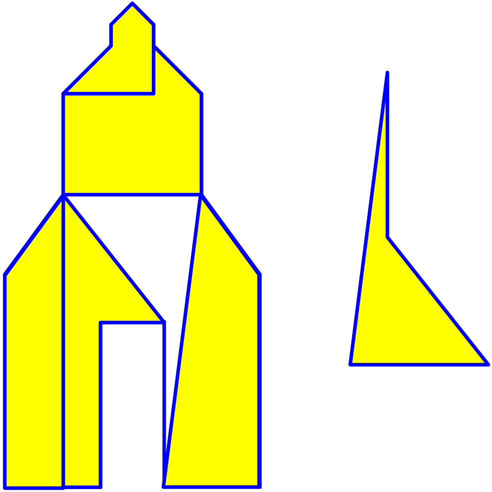
Tangrams:





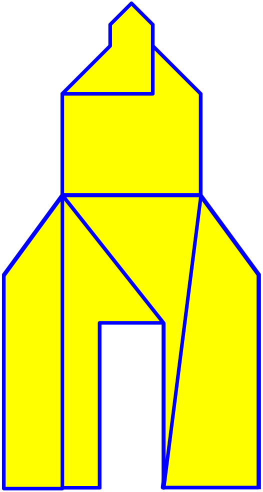
What is a tiling?

Tangrams:



What is a tiling?

Tangrams:



Tilings

- No overlap allowed
- Must completely cover the region

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

The kind of questions a mathematician might ask:

- Is there a tiling with the given pieces?

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

The kind of questions a mathematician might ask:

- Is there a tiling with the given pieces?
- Is it easy to prove that a tiling doesn't exist?

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

The kind of questions a mathematician might ask:

- Is there a tiling with the given pieces?
- Is it easy to prove that a tiling doesn't exist?
- How many tilings are there?

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

The kind of questions a mathematician might ask:

- Is there a tiling with the given pieces?
- Is it easy to prove that a tiling doesn't exist?
- How many tilings are there?
- What does a typical tiling look like?

Tilings

- No overlap allowed
- Must completely cover the region: area of region equals the sum of the areas of the tiles.

Applications:

- Archaeology: reassembling fragments.
- Packing: loading trucks, allocating bandwidth, scheduling airline flights.

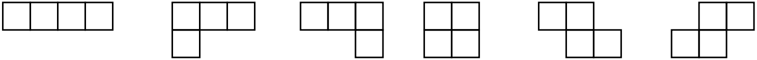
The kind of questions a mathematician might ask:

- Is there a tiling with the given pieces?
- Is it easy to prove that a tiling doesn't exist?
- How many tilings are there?
- What does a typical tiling look like?

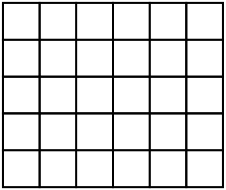
Based on an expository paper of Richard Stanley and Federico Ardila.

Is there a tiling?

Tetris pieces:

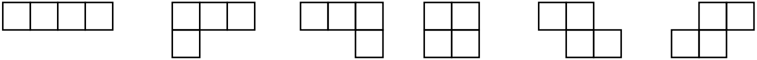


Can we tile a 6×5 rectangle with the tetris pieces, using each piece as many times as we like?

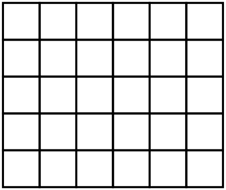


Is there a tiling?

Tetris pieces:



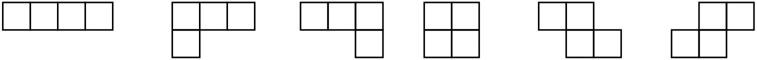
Can we tile a 6 $\times$ 5 rectangle with the tetris pieces, using each piece as many times as we like?



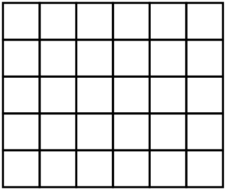
No.

Is there a tiling?

Tetris pieces:



Can we tile a 6×5 rectangle with the tetris pieces, using each piece as many times as we like?

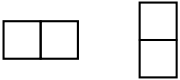


No.

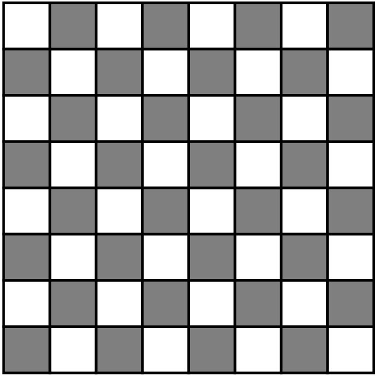
Each piece has 4 boxes.
There are 30 boxes to fill.
4 does not divide into 30 evenly. (Divisibility argument)

Is there a tiling of a chessboard with dominoes?

Dominoes:

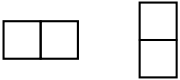


Can we tile a chessboard with dominoes?
64 squares.



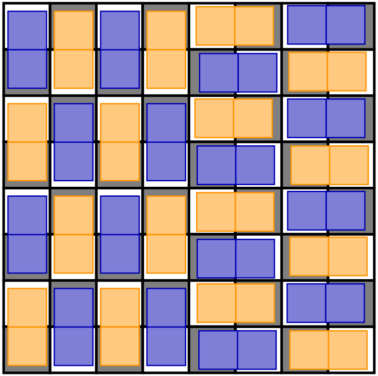
Is there a tiling of a chessboard with dominoes?

Dominoes:



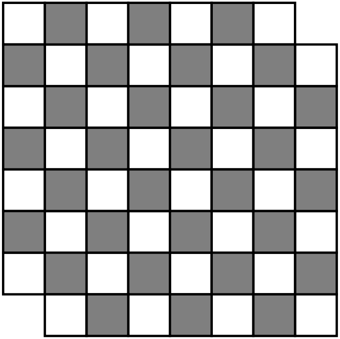
Can we tile a chessboard with dominoes? Yes.

64 squares.



Is there a tiling of a holey chessboard?

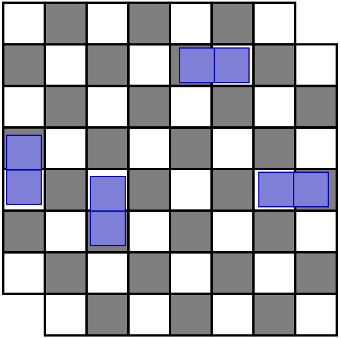
Can we tile a this modified chessboard with dominoes?
62 squares: 30 black, 32 white.



Is there a tiling of a holey chessboard?

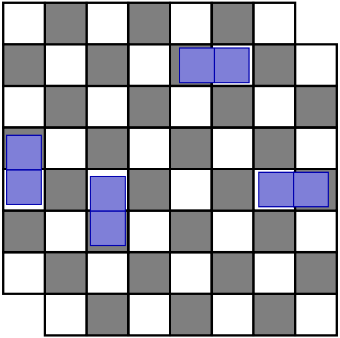
Can we tile a this modified chessboard with dominoes? No.

62 squares: 30 black, 32 white.



Is there a tiling of a holey chessboard?

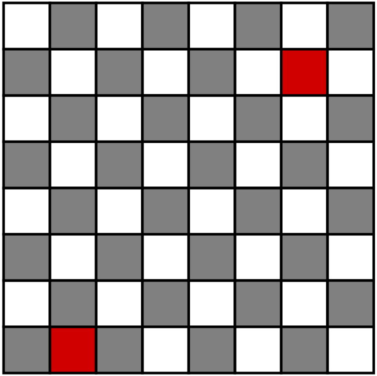
Can we tile a this modified chessboard with dominoes? No.
62 squares: 30 black, 32 white.



Every domino covers exactly one black square and one white square.
But there are not the same number of white squares as black squares. (Coloring argument)

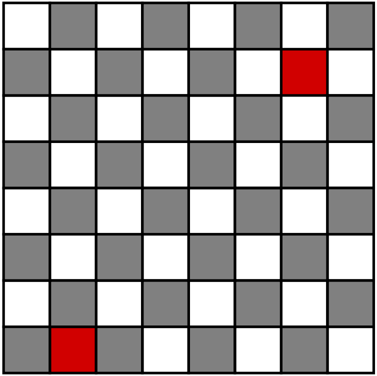
Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square?
62 squares: 31 black, 31 white.



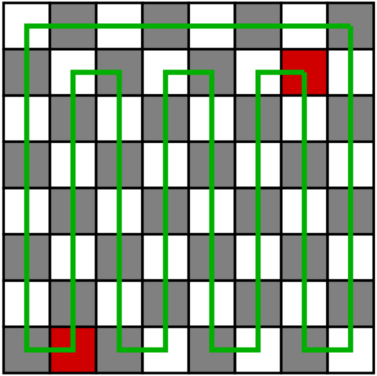
Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square? Yes.
62 squares: 31 black, 31 white.



Is there a tiling of a fair holey chessboard?

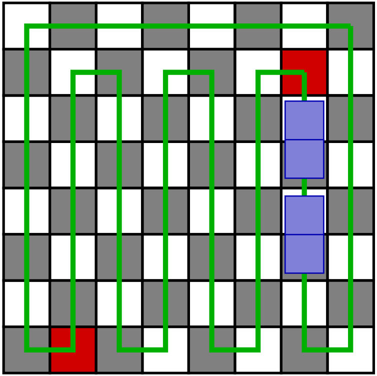
What if we remove 1 black and 1 white square? Yes.
62 squares: 31 black, 31 white.



Is there a tiling of a fair holey chessboard?

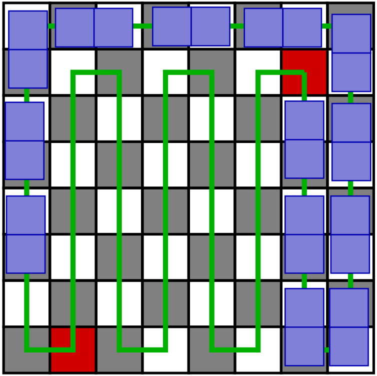
What if we remove 1 black and 1 white square? Yes.

62 squares: 31 black, 31 white.



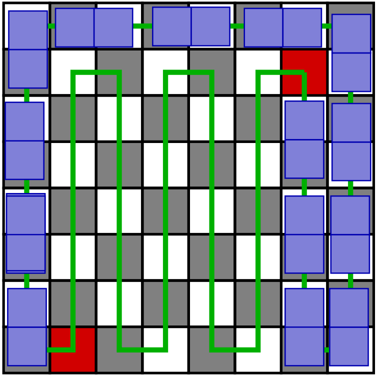
Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square? Yes.
62 squares: 31 black, 31 white.



Is there a tiling of a fair holey chessboard?

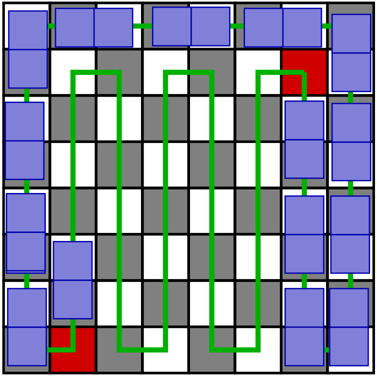
What if we remove 1 black and 1 white square? Yes.
62 squares: 31 black, 31 white.



Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square? Yes.

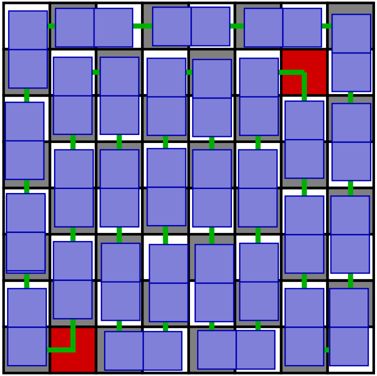
62 squares: 31 black, 31 white.



Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square? Yes.

62 squares: 31 black, 31 white.

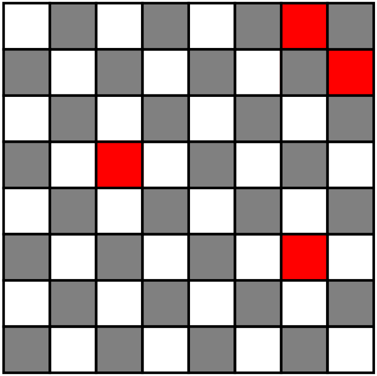


Is there a tiling of a fair holey chessboard?

What if we remove any 2 black and any 2 white squares?
60 squares: 30 black, 30 white.

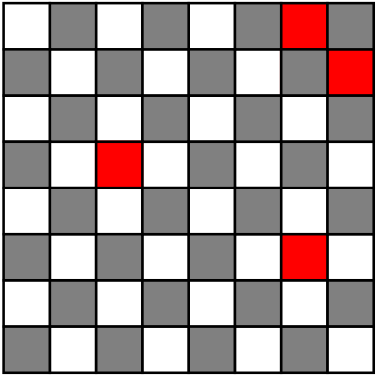
Is there a tiling of a fair holey chessboard?

What if we remove any 2 black and any 2 white squares? No.
60 squares: 30 black, 30 white.



Is there a tiling of a fair holey chessboard?

What if we remove any 2 black and any 2 white squares? No.
60 squares: 30 black, 30 white.



Question
What if the holey chessboard has to be connected?

Demonstrating that a tiling does not exist

If a tiling of a region exists: easy to demonstrate.

Question

If a tiling doesn't exist, is there an "easy" way to demonstrate that?

Demonstrating that a tiling does not exist

If a tiling of a region exists: easy to demonstrate.

Question

If a tiling doesn't exist, is there an "easy" way to demonstrate that?

No, in general.
Beauquier, Nivat, Rémila and Robson (1995):
for $1 \times n$ tiles with $n > 2$, determining if a region can be tiled is an *NP-complete* problem.

Demonstrating that a tiling does not exist

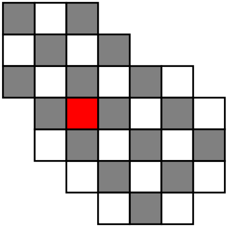
If a tiling of a region exists: easy to demonstrate.

Question

If a tiling doesn't exist, is there an "easy" way to demonstrate that?

No, in general.
Beauquier, Nivat, Rémila and Robson (1995):
for $1 \times n$ tiles with $n > 2$, determining if a region can be tiled is an *NP-complete* problem.

Yes, for domino tilings.



Demonstrating that a tiling does not exist

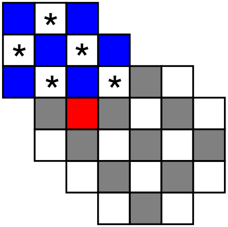
If a tiling of a region exists: easy to demonstrate.

Question

If a tiling doesn't exist, is there an "easy" way to demonstrate that?

No, in general.
Beauquier, Nivat, Rémila and Robson (1995):
for $1 \times n$ tiles with $n > 2$, determining if a region can be tiled is an *NP-complete* problem.

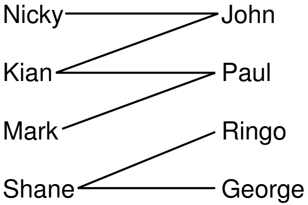
Yes, for domino tilings.



Hall's Marriage Theorem

Theorem (Hall's Marriage Theorem, 1935)

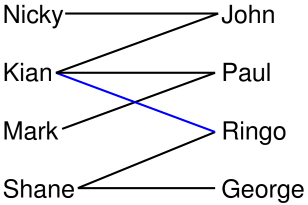
n women, n men.
Each woman W, compatible husbands S_W.
Perfect matching exists if and only if:
for all i and for every subset of i women, the union of the corresponding S_W has size at least i.



Hall's Marriage Theorem

Theorem (Hall's Marriage Theorem, 1935)

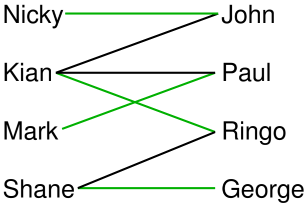
n women, n men.
Each woman W, compatible husbands S_W.
Perfect matching exists if and only if:
for all i and for every subset of i women, the union of the corresponding S_W has size at least i.



Hall's Marriage Theorem

Theorem (Hall's Marriage Theorem, 1935)

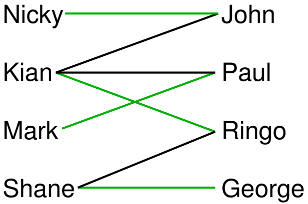
n women, n men.
Each woman W, compatible husbands S_W.
Perfect matching exists if and only if:
for all i and for every subset of i women, the union of the corresponding S_W has size at least i.



Hall's Marriage Theorem

Theorem (Hall's Marriage Theorem, 1935)

n women, n men.
Each woman W, compatible husbands S_W.
Perfect matching exists if and only if:
for all i and for every subset of i women, the union of the corresponding S_W has size at least i.



Application

Women: black squares.
Men: white squares.
Tiling $\iff$ perfect matching.
No tiling: some subset of black squares which shows this.

How many tilings of a chessboard with dominoes?

How many tilings of a chessboard with dominoes?

Fisher & Temperley, Kasteleyn (independently, 1961):
The number of tilings of a $2m \times 2n$ rectangle with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

For example, for a chessboard $m = n = 4$, and we get

$$4^{16} \prod_{j=1}^{4} \prod_{k=1}^{4} \left(\cos^2 \frac{j\pi}{9} + \cos^2 \frac{k\pi}{9} \right).$$

How many tilings of a chessboard with dominoes?

Fisher & Temperley, Kasteleyn (independently, 1961):
The number of tilings of a $2m \times 2n$ rectangle with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

For example, for a chessboard $m = n = 4$, and we get

$$4^{16} \prod_{j=1}^{4} \prod_{k=1}^{4} \left(\cos^2 \frac{j\pi}{9} + \cos^2 \frac{k\pi}{9} \right).$$

This is an amazing formula!
e.g. $\cos^2 20^\circ = 0.8830222216\ldots$.

How many tilings of a chessboard with dominoes?

Fisher & Temperley, Kasteleyn (independently, 1961):
The number of tilings of a $2m \times 2n$ rectangle with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

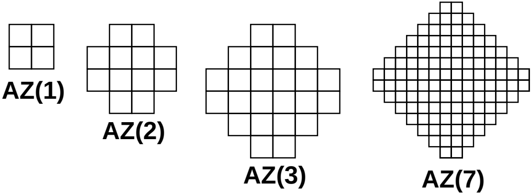
For example, for a chessboard $m = n = 4$, and we get

$$4^{16} \prod_{j=1}^{4} \prod_{k=1}^{4} \left(\cos^2 \frac{j\pi}{9} + \cos^2 \frac{k\pi}{9} \right).$$

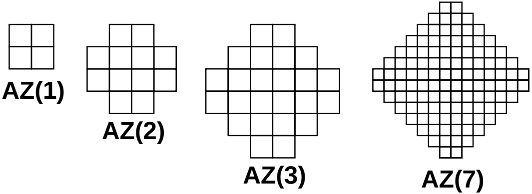
This is an amazing formula!
e.g. $\cos^2 20^\circ = 0.8830222216\ldots$.

Answer = 12,988,816 .

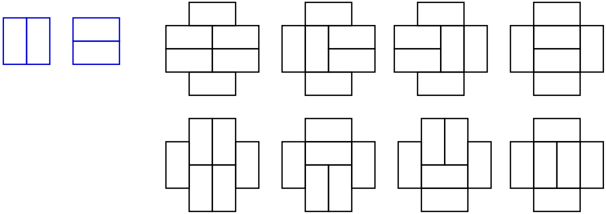
How many tilings of Aztec diamonds with dominoes?



How many tilings of Aztec diamonds with dominoes?



Tilings with dominoes:



How many tilings of Aztec diamonds (continued)

$2, 8, 64, 1024, \ldots.$

Elkies, Kuperberg, Larsen & Propp (1992):
In general, AZ(n) has $2^{\frac{n(n+1)}{2}}$ tilings with dominoes. (4 proofs)

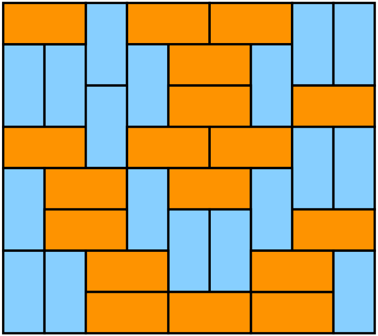
Now around 12 proofs, but none are really simple.

Open Problem

Find a simple proof that the number of tilings of AZ(n) is $2^{\frac{n(n+1)}{2}}$*.*

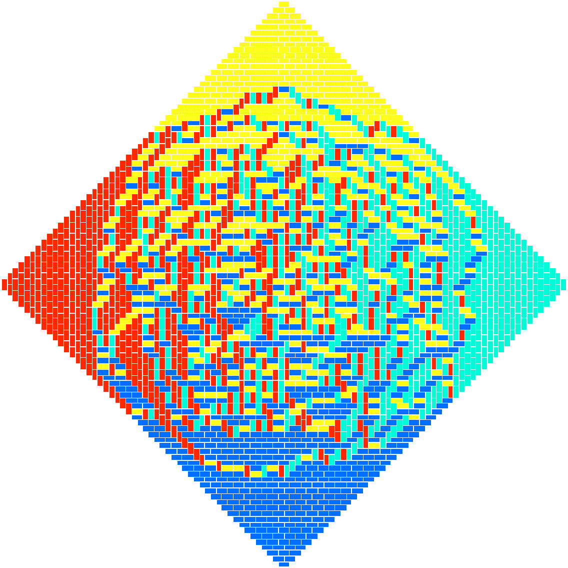
$$2^{\frac{n(n+1)}{2}} = 2^{\binom{n+1}{2}} = 2^{1+2+\cdots+n}$$

What does a typical tiling look like?



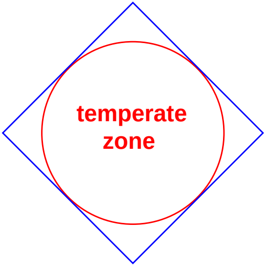
No obvious structure.
But if we work with Aztec diamonds....

A typical tiling of AZ(50)



Tilings and global warming

Jockusch, Propp and Shor (1995).
The Arctic Circle Theorem. Fix $\varepsilon > 0$. Then for all sufficiently large *n*, all but an ε fraction of the domino tilings of AZ(*n*) will have a temperate zone whose boundary stays uniformly within distance εn of the inscribed circle.



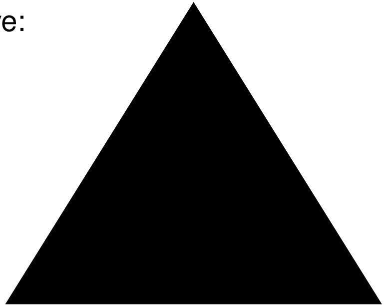
In other words: almost everything outside and not too close to the circle is “frozen” in place.

Similar phenomena observed for other cases.

“To infinity and beyond” – Lightyear, Buzz (1995)

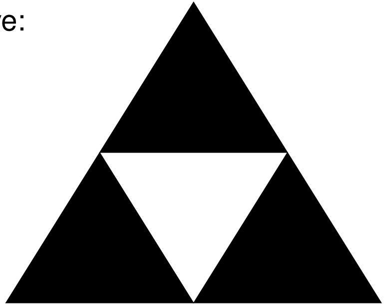
“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



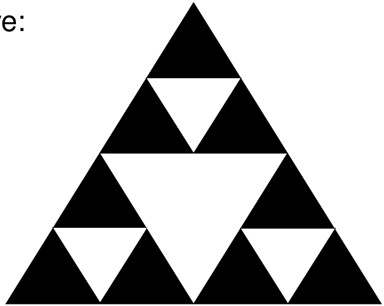
“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set

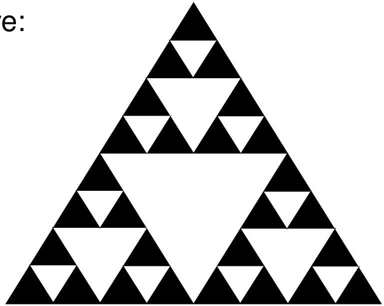


“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set

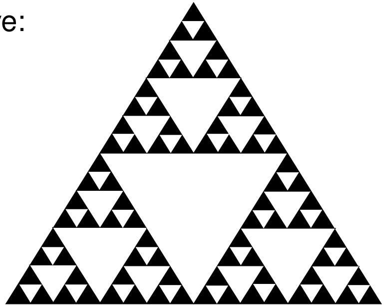


Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



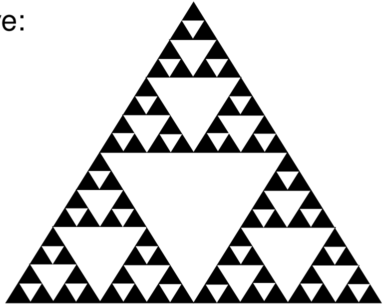
“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



"To infinity and beyond" – Lightyear, Buzz (1995)

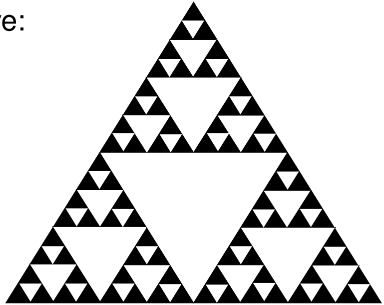
Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



Area of black portion $= 1 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdots = 0.$

“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set

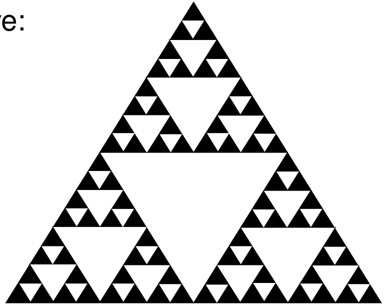


$$\text{Area of black portion } = 1 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdots = 0.$$

Conclusion: in the limit, the white triangles tile the big triangle.

“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



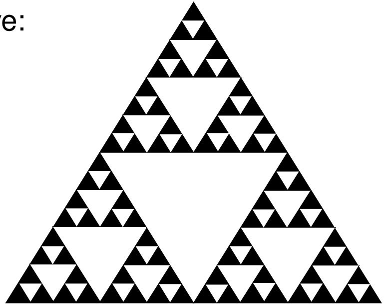
$$\text{Area of black portion } = 1 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdots = 0.$$

Conclusion: in the limit, the white triangles tile the big triangle.

$$\text{Area of white potion} \quad = \quad \frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \cdots$$

“To infinity and beyond” – Lightyear, Buzz (1995)

Sierpinski triangle/gasket/sieve:
like a 2-dim verison
of the Cantor Set



$$\text{Area of black portion } = 1 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdots = 0.$$

Conclusion: in the limit, the white triangles tile the big triangle.

$$\begin{aligned} \text{Area of white potion} &= \frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \cdots \\ &= \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 1. \end{aligned}$$

Sierpinski triangle side comment

The Sierpinski triangle is very fashionable:

Sierpinski triangle side comment

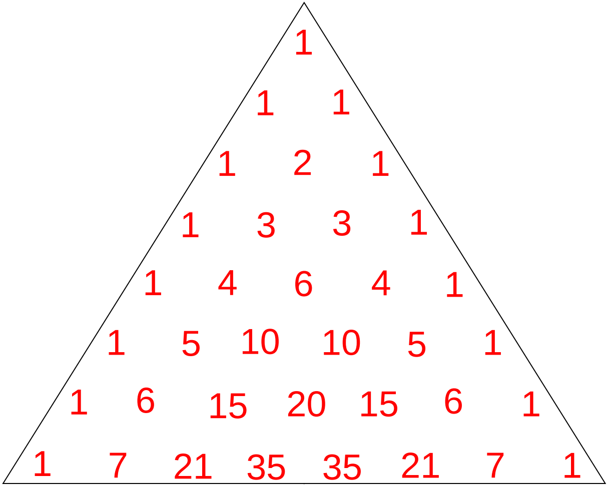
The Sierpinski triangle is very fashionable:

Designer: Eri Matsui

Another Sierpinski triangle side comment

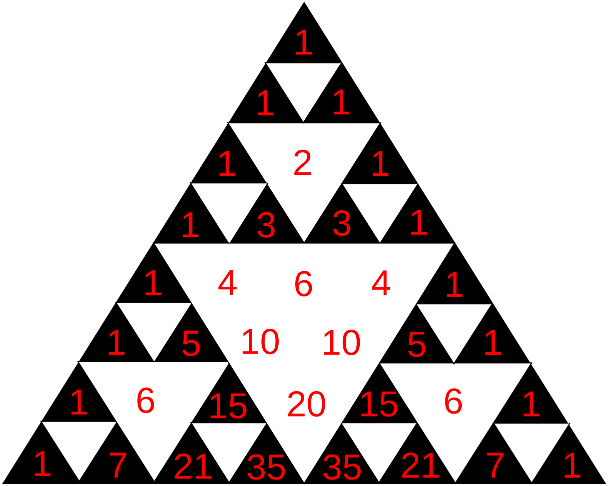
Another famous triangle is Pascal's triangle.

Take the first 2^n rows:



Another Sierpinski triangle side comment

Another famous triangle is Pascal's triangle.
Take the first 2^n rows:



From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)$$

From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)$$

From a series to a tiling

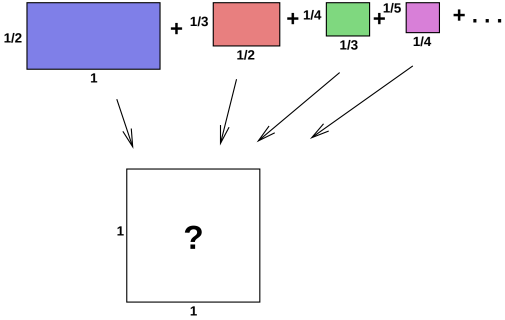
$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$

From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

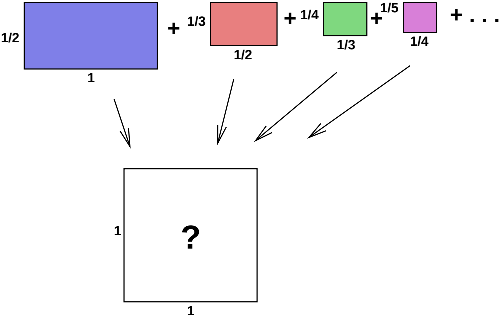
$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$

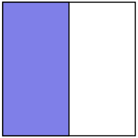


From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$

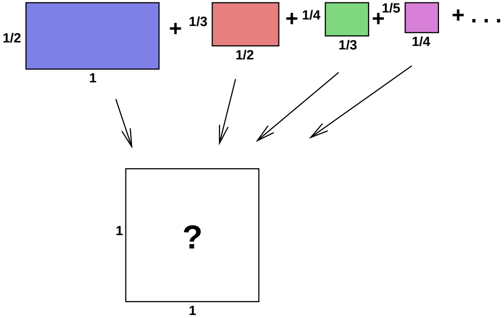


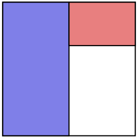


From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$

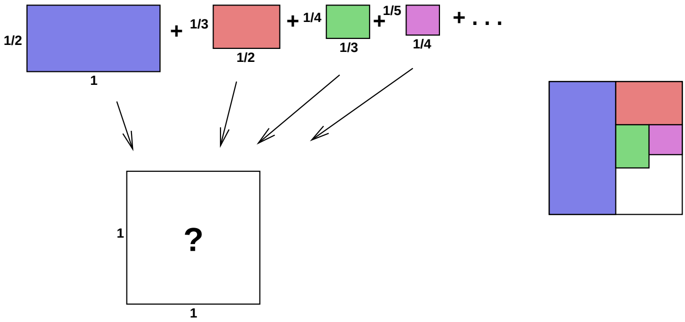




From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

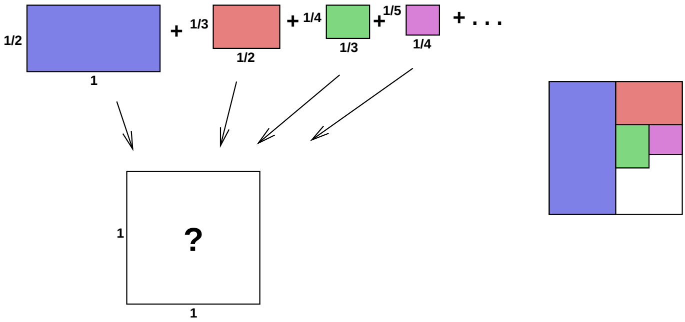
$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$



From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$



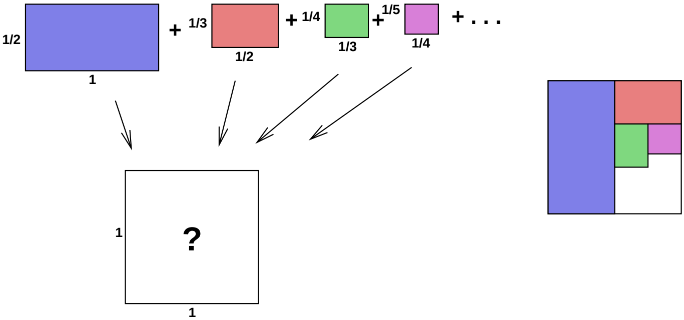
Open Problem

Find a way to tile the whole region, or show that no tiling exists.

From a series to a tiling

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$=\ \sum_{n=1}^{\infty}\left(\frac{A}{n}+\frac{B}{n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{-1}{n+1}\right)=1.$$



Open Problem

Find a way to tile the whole region, or show that no tiling exists.

Paulhus (1998): side length 1.000000001

Tiling infinite regions

Alhambra palace, Granada, Spain.

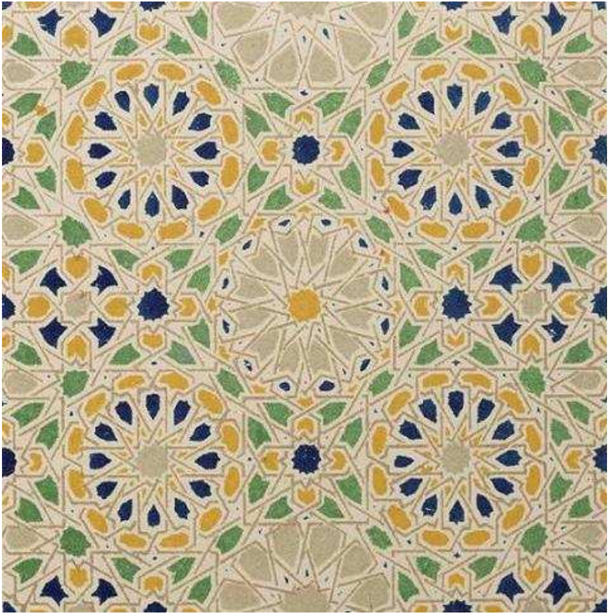
Tiling infinite regions

Alhambra palace, Granada, Spain.

Abstract Algebra: There are essentially 17 different tiling patterns of the plane that have translation symmetries in two different directions.

Plane crystallographic groups / wallpaper groups

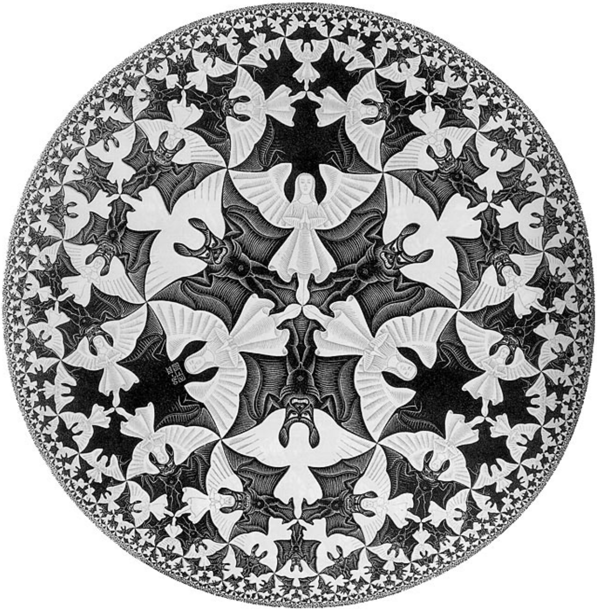
Another Alhambra tiling



Escher tilings

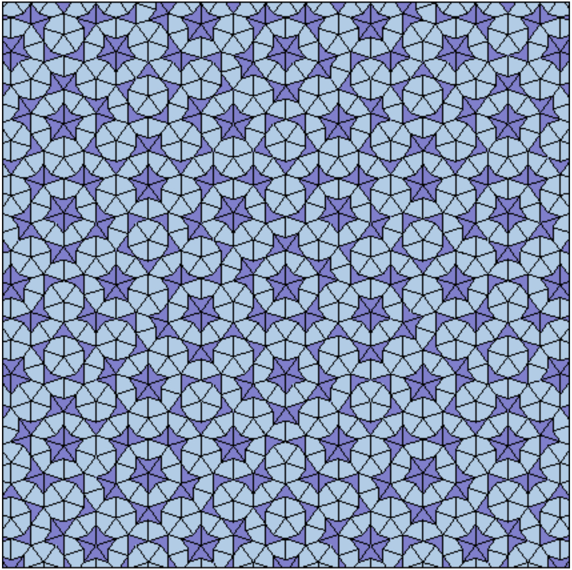
Maurits Cornelis Escher (1898-1972): *Although I am absolutely without training in the exact sciences, I often seem to have more in common with mathematicians that with my fellow artists.*

Another Escher tiling



Opposite direction: no symmetry at all!

Sir Roger Penrose



Another Penrose tiling