From Dyck Paths to Standard Young Tableaux

Peter McNamara
Bucknell University & LaBRI

Joint work with Juan Gil, Jordan Tirrell, and Michael Weiner

GT Combinatoire Énumérative et Algébrique
30 September 2019

Slides and paper available from
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Jordan helped with slides

Context

Much of my research:

Symmetric functions
Schur functions s_λ

Quasisymmetric functions
F-expansions
(Gessel's fundamental basis)

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Lattice paths

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Well understood:
Dyck paths

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Standard Young Tableaux (SYT) $\longleftrightarrow$ Well understood: Dyck paths

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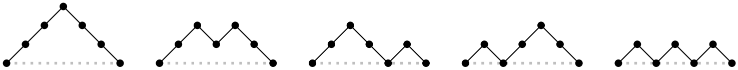
Outline

- Background, main question, classic example
- Hook shapes and flag shapes
- A much more elaborate example

Definition. A **Dyck path of semilength n** is a sequence of up steps $\mathrm{U} = (1, 1)$ and down steps $\mathrm{D} = (1, -1)$ from $(0, 0)$ to $(2n, 0)$ that stays weakly above the x-axis.

Example. The five Dyck paths of semilength 3.



The number of Dyck paths of semilength n is the Catalan number C_n.

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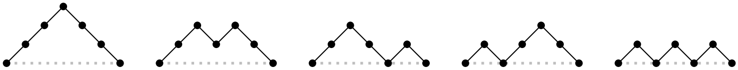


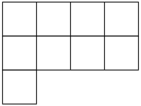
The number of Dyck paths of semilength n is the Catalan number C_n.

Definition. An ascent of a Dyck path is a maximal consecutive sequence of up-steps, and it is a k-ascent if it has length k.

Definitions: standard Young tableaux

Definition. For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, a Young diagram of shape λ is an array of boxes left- and top-justified with λ_i boxes in row i.

Example. $\lambda = (4, 4, 1)$



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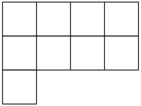

Definition. A standard Young tableau or SYT is a Young diagram whose n boxes are filled bijectively with $\{1, \ldots, n\}$ such that the entries increase along rows and down columns.

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Definition. A standard Young tableau or SYT is a Young diagram whose n boxes are filled bijectively with $\{1, \ldots, n\}$ such that the entries increase along rows and down columns.

The number of SYT of shape λ is given by the hook-length formula.

The main question

In what ways can we add extra structure or restrictions to Dyck paths and/or SYT to yield equinumerous sets?

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We give 4 ways to answer this question:

0. the classic bijection;
1,2. a pair with the same format;
3. an elaborate bijection.

Bijection 0. The classic example

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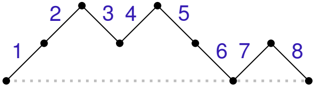
Theorem. Dyck paths of semilength n are in bijection with the SYT of shape (n, n).

Proof. Put indices of U steps in the first row and indices of D steps in the second row.

Example.



$\longleftrightarrow$

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Note. #boxes = 2(semilength).
In our other examples,
#boxes = semilength.

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Dyck paths $\longleftrightarrow$ nomincreasing set partitions $\longleftrightarrow$ modified tableaux

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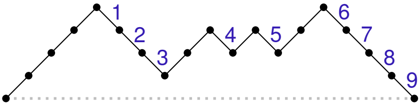
To label U steps:

1. Label the D steps $1, \ldots, n$ from left-to-right.
2. At each peak UD, give the U the same label as the D.
3. Going through the ascents from left-to-right, label the remaining U in a greedy fashion from top-to-bottom.



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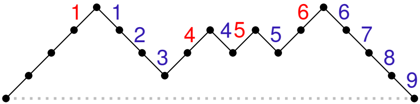

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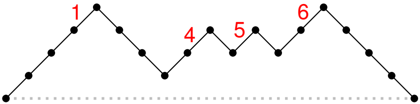

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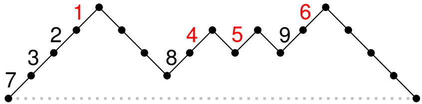

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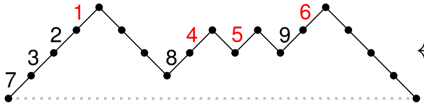

$\longleftrightarrow$ 1237|48|5|69

- Nomincreasing (set) partitions: in standard form, non-minimum entries in each block form an increasing sequence: 23789.

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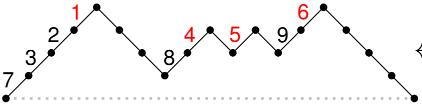

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- Nomincreasing (set) partitions: in standard form, non-minimum entries in each block form an increasing sequence: 23789.
- Modified tableaux: entries increase along first row and down columns; non-first-row entries increase left-to-right.

The main question

Recall the main question:

In what ways can we add extra structure or restrictions to Dyck paths and/or SYT to yield equinumerous sets?

Want bijective proofs that preserve some statistics.

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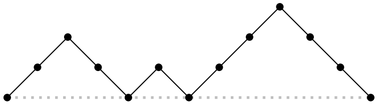
Bijection 1. Hook shapes

Baby Theorem. For $1 \leq k \leq n$, Dyck paths of semilength n with k peaks and k returns are in bijection with SYT of hook shape $(k, 1^{n-k})$.

(1^{n-k} denotes a sequence of $n - k$ copies of 1.)



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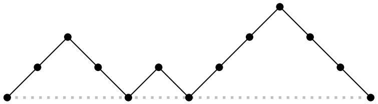

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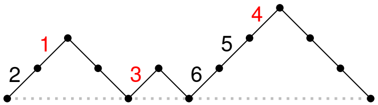

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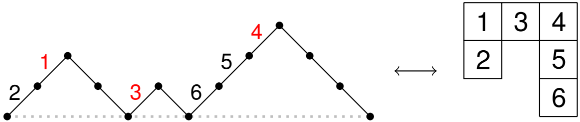

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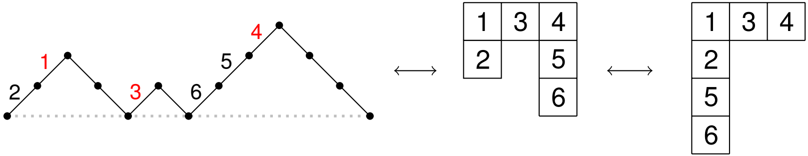

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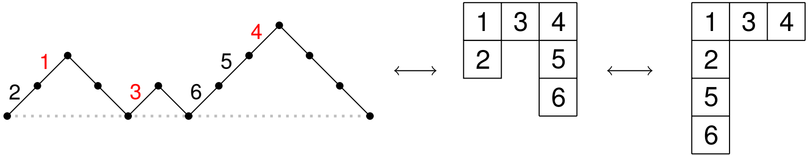

Main idea for inverse direction: In this special situation, the columns of the modified tableau have increasing **consecutive** entries.

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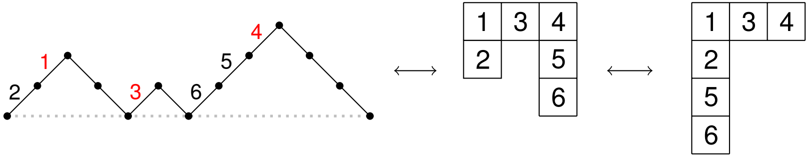

Main idea for inverse direction: In this special situation, the columns of the modified tableau have increasing **consecutive** entries.

Corollary. The number of Dyck paths of semilength n with as many peaks as returns equals the number of SYT of hook shape with n boxes.

Bijection 2: Flag shapes

Definition. AN SYT is of flag shape if its shape is $(k, k, 1^{n-2k})$ for some $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

1	3	4	5	9	10	16
2	7	12	13	14	15	17
6						
8						
11						

Definition. An ascent is a singleton if it has length 1.

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Definition. An ascent is a singleton if it has length 1.

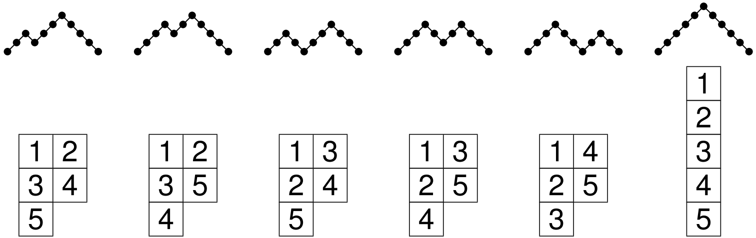
Theorem. The number of Dyck paths of semilength n and no singletons equals the number of SYT of flag shape with n boxes.

These sets are enumerated by the Riordan numbers [A005043].

Bijection 2: Flag shapes

Theorem. The number of Dyck paths of semilength n without singletons equals the number of SYT of flag shape with n boxes.

Example. Let $n = 5$.



Bijection 2: Flag shapes

Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

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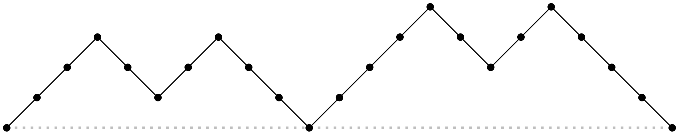
Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

Proof. By defining modified tableaux, we've done the hard part.



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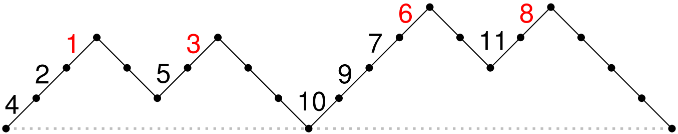

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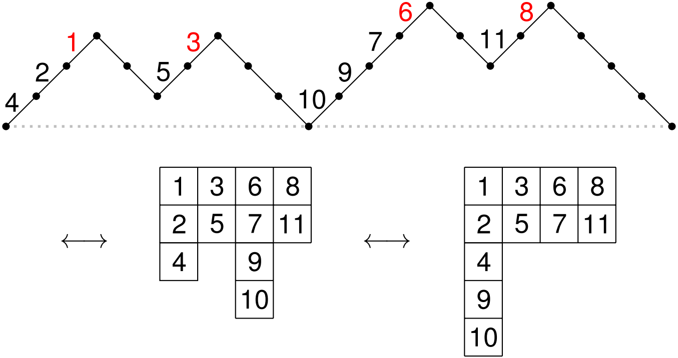

First two rows are fixed since there are no singletons.
For inverse, use: non-first-row entries increase from left-to-right.

Remarks on Bijection 2

Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

Corollary. The number of Dyck paths of semilength n without singletons equals the number of SYT of flag shape with n boxes.

Letting $n = 2k$ in the theorem:
SYT of shape (k, k)
$\longleftrightarrow$ Dyck paths of semilength k (by Bijection 0)
$\Rightarrow C_k$

Observation. The Catalan number C_k counts the number of Dyck paths of semilength $2k$ with k peaks and no singletons.

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Also uses modified tableaux; rest of bijection is intricate.

Theorem. The number of Dyck paths of semilength n that avoid three consecutive up-steps equals the number of SYT with n boxes and at most 3 rows.

Bijection 3: All SYT

What if we want a bijection to all SYT?

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Answer 1 [Françon and Viennot]. Height-labeled Motzkin paths of length n with s flat steps are in bijection with SYT with n boxes and s odd columns.

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Answer 2 [GMTW and Burrill–Courtiel–Fusy–Melczer–Mishna]. Use cm-labeled Dyck paths.

What is a cm-labeled Dyck path?

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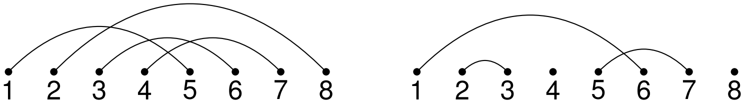
What is a cm-labeled Dyck path?

Theorem. The number of cm-labeled Dyck paths of semilength n equals the number of SYT with n boxes.

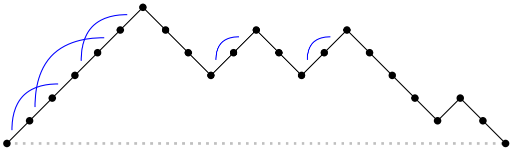
Theorem. The number of cm-labeled Dyck paths of semilength n with s singletons and k-noncrossing labels equals the number of SYT with n boxes, s odd columns, and at most $2k - 1$ rows.

cm-labeled Dyck paths

Definition. A partial matching is connected if the arcs and points form a connected set as a subset of the plane.



Definition. A cm-labeled Dyck path is a Dyck path where each k-ascent is labeled by a connected matching of $[k]$, for every k.

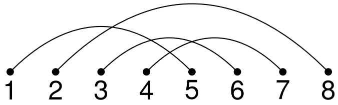


Note. This is both a restriction and additional structure on Dyck paths (ascents lengths must be one or even, but ascents with length at least six have multiple possible labels).

k-noncrossing and k-nonnesting

Theorem. The number of cm-labeled Dyck paths of semilength n with s singletons and k-noncrossing labels equals the number of SYT with n boxes, s odd columns, and at most $2k - 1$ rows.

Definition. A k-crossing is a set of k arcs in a partial matching that are pairwise crossing.
We say a partial matching is k-noncrossing if it has no k-crossings.
Similarly for k-nesting and k-nonnesting.



The matching (15)(28)(36)(47) has a 3-crossing (15)(36)(47) but is 4-noncrossing.
It has a 2-nesting (28)(36) or (28)(47) but is 3-nonnesting.

k-noncrossing and *k*-nonnesting

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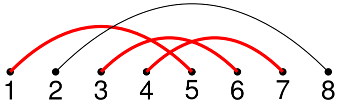

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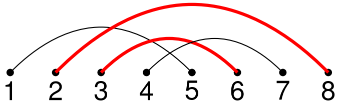

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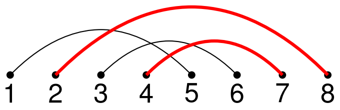
k-noncrossing and k-nonnesting

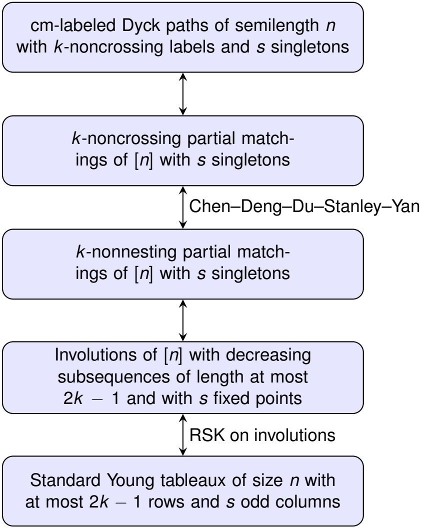
Theorem. The number of cm-labeled Dyck paths of semilength n with s singletons and k-noncrossing labels equals the number of SYT with n boxes, s odd columns, and at most $2k-1$ rows.

Definition. A k-crossing is a set of k arcs in a partial matching that are pairwise crossing.
We say a partial matching is k-noncrossing if it has no k-crossings.
Similarly for k-nesting and k-nonnesting.



The matching (15)(28)(36)(47) has a 3-crossing (15)(36)(47) but is 4-noncrossing.
It has a 2-nesting (28)(36) or (28)(47) but is 3-nonnesting.

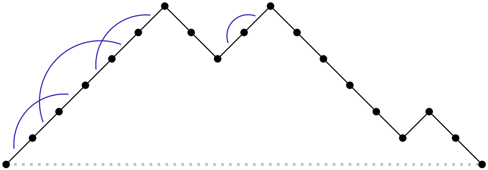
Structure of the bijection



Bijectivity among bottom 4 blocks appears in Burrill–Courtiel–Fusy–Melczer–Mishna.

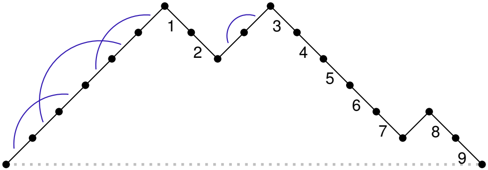
cm-labeled Dyck paths to partial matchings

1. Start with a cm-labeled Dyck path



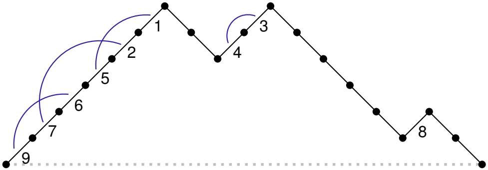
cm-labeled Dyck paths to partial matchings

1. Start with a cm-labeled Dyck path
2. Label the down steps from left-to-right



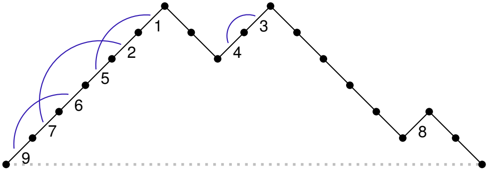
cm-labeled Dyck paths to partial matchings

1. Start with a cm-labeled Dyck path
2. Label the down steps from left-to-right
3. Match down steps to up steps horizontally



cm-labeled Dyck paths to partial matchings

1. Start with a cm-labeled Dyck path
2. Label the down steps from left-to-right
3. Match down steps to up steps horizontally
4. The ascents form a non-crossing set partition of [n]: 125679|34|8



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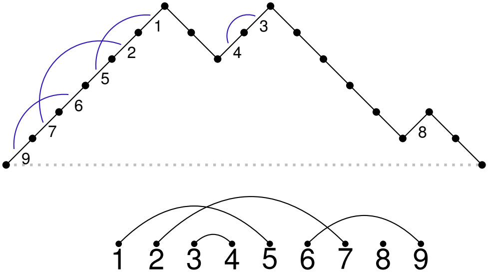
5. Add cm labels to get a partial matching



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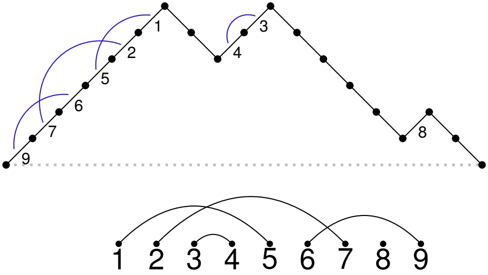

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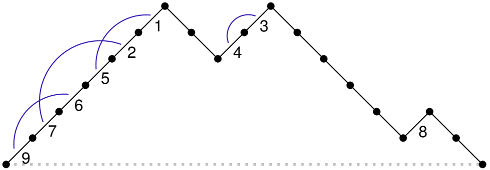
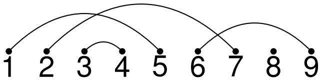
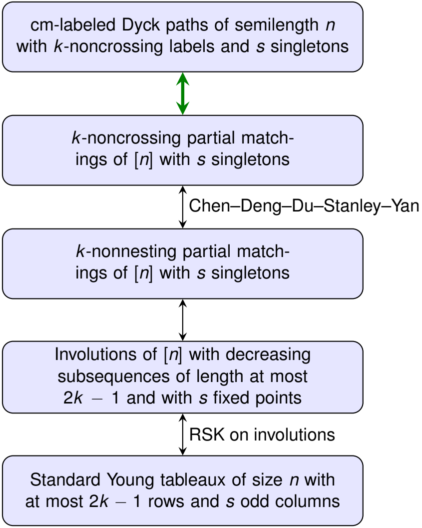




Note. Crossings and singletons preserved.

Inverse: Connected components give ascents. Steps 2–4 give a well-known bijection from unlabeled Dyck paths to non-crossing set partitions.

Structure of the bijection



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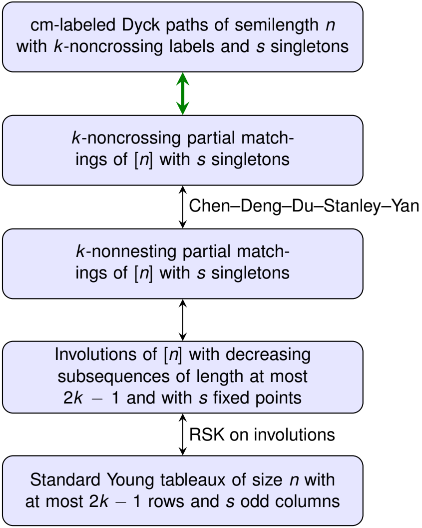
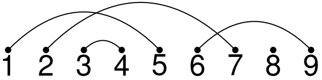


Next: bottom bijection.

Involutions to SYT

First observation. Partial matchings are in bijection with involutions (self-inverse permutations):



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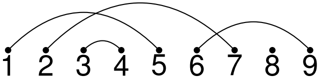

$\longleftrightarrow \quad (15)(27)(34)(69)(8).$

Robinson–Schensted–Knuth (RSK) Algorithm.

$$\text{permutation} \quad \pi \longleftrightarrow (T, R) \quad \text{two SYT of same shape.}$$

Robinson, Schützenberger: $\pi^{-1} \longleftrightarrow (R, T)$.

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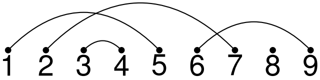

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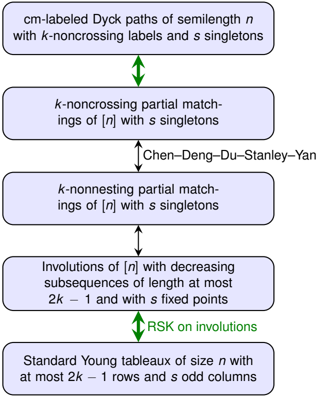
So if π is an involution, $\pi \longleftrightarrow (T, T) \longleftrightarrow T$.

Other facts we need:

- Knuth: # fixed points (singletons) in π = # odd columns in T.
- Schensted:
  Length of longest decreasing subsequence in π = # rows in T.

Structure of the bijection



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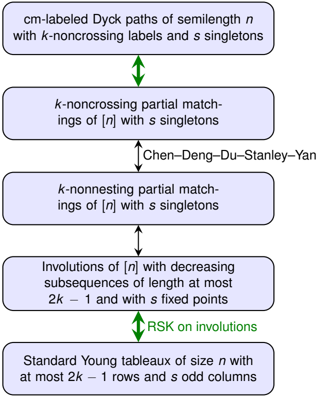

We have:

cm-labeled Dyck paths $\longleftrightarrow$ partial matchings $\longleftrightarrow$ involutions $\longleftrightarrow$ SYT.

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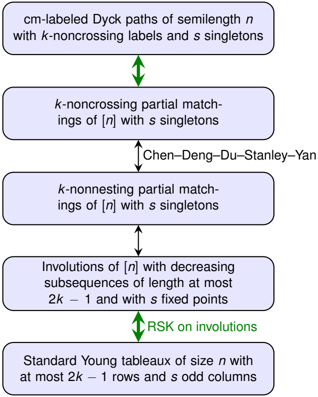

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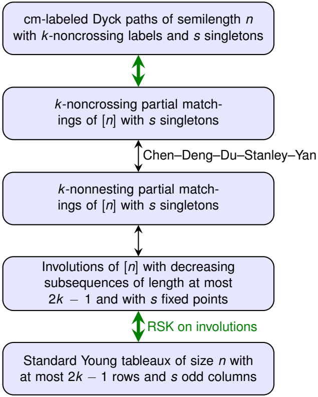
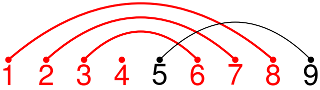
Difficulty. No connection between **crossings** and decreasing subseqences.
Nice connection between **nestings** and decreasing subseqences.
Next: k-nesting $\iff$ a decreasing subsequence of length at least $2k$.

Structure of the bijection





$\longrightarrow \quad \pi = 876493215$

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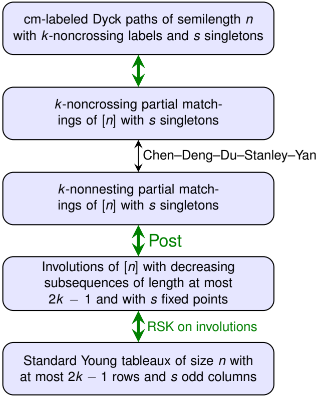

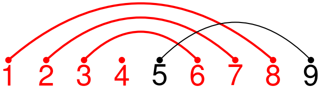

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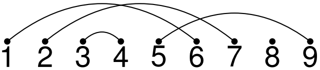
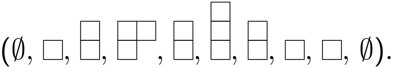
Final step. A bijection from k-noncrossing to k-nonnesting partial matchings of $[n]$ (which preserves singletons).
Chen–Deng–Du–Stanley–Yan: use oscillating tableaux.
We need to use weakly oscillating tableaux.

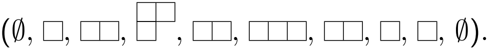
Overview of proof by example. Map the partial matching



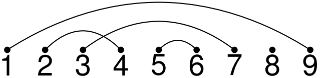
to the weakly oscillating tableau



Take the **transpose**:

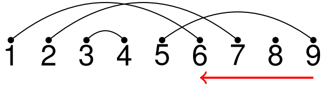


and reverse the map:



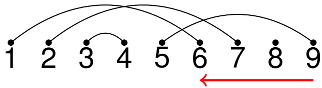
The point. k-crossing $\longleftrightarrow$ k-nesting.

Example details.

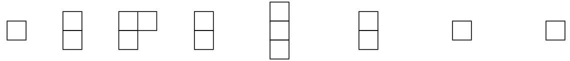
$M =$ 

j	0	1	2	3	4	5	6	7	8	9
T^j	$\emptyset$	1	1 2	1 3 2	1 2	1 2 5	2 5	5	5	$\emptyset$

Example details.

$M =$ 

j	0	1	2	3	4	5	6	7	8	9
T^j	∅	1	1 / 2	1 3 / 2	1 / 2	1 / 2 / 5	2 / 5	5	5	∅ ←
λ^j	∅									∅



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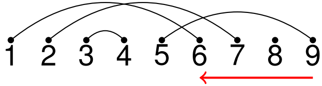
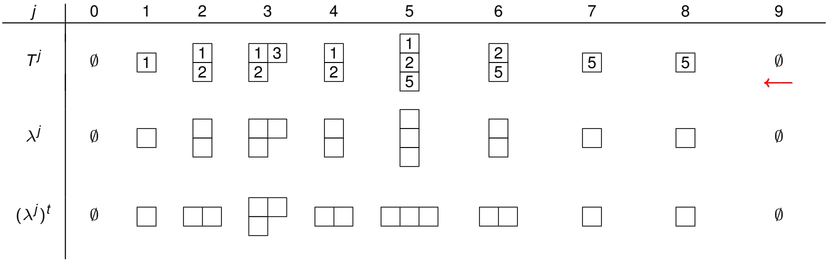
$M =$ 



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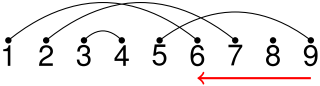

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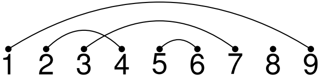
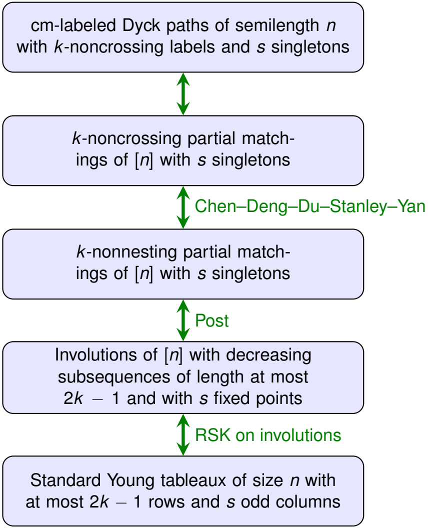
T^j	$\emptyset$	1	1 2	1 3 2	1 2	1 2 5	2 5	5	5	$\emptyset$
λ^j	$\emptyset$	□	□ □	□□ □	□ □	□ □ □	□ □	□	□	$\emptyset$
$(\lambda^j)^t$	$\emptyset$	□	□□	□□ □	□□	□□□	□□	□	□	$\emptyset$
$\widehat{T}^j$	$\emptyset$	1	1 2	1 2 3	1 3	1 3 5	1 3	1	1	$\emptyset$
$\widehat{M}^j$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	(2, 4)	(2, 4)	(2, 4) (5, 6)	(2, 4) (5, 6) (3, 7)	(2, 4) (5, 6) (3, 7)	(2, 4) (5, 6) (3, 7) (1, 9)

$\widehat{M} =$



The end


cm-labeled Dyck paths of semilength n with k-noncrossing labels and s singletons
k-noncrossing partial matchings of $[n]$ with s singletons
Chen–Deng–Du–Stanley–Yan
k-nonnesting partial matchings of $[n]$ with s singletons
Post
Involutions of $[n]$ with decreasing subsequences of length at most $2k - 1$ and with s fixed points
RSK on involutions
Standard Young tableaux of size n with at most $2k - 1$ rows and s odd columns

The end

cm-labeled Dyck paths of semilength n with k-noncrossing labels and s singletons

↕

k-noncrossing partial matchings of $[n]$ with s singletons

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MERCI!