A Pieri rule for skew shapes

Peter McNamara
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Joint work with:
Sami Assaf
MIT

4 August 2010

Full paper available from
`www.facstaff.bucknell.edu/pm040/`

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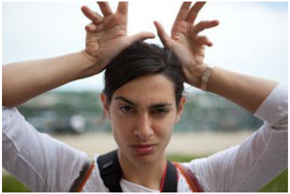

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Outline

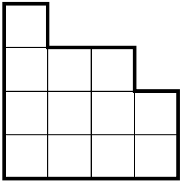
- Background on skew Schur functions and Pieri rule
- Main result
- Some highlights of the combinatorial proof
- 3 further-development nuggets

Schur functions

Cauchy, 1815

- Partition
  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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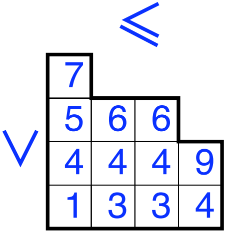
- Semistandard Young tableau (SSYT)



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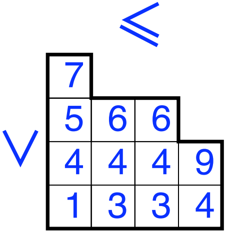


The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

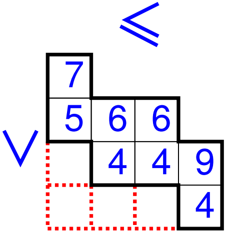
Example:

$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

Cauchy, 1815

- Partition
  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ
- Young diagram
  Example:
  $\lambda/\mu = (4, 4, 3, 1)/(3, 1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

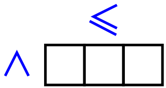
$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example:

$s_{4431/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

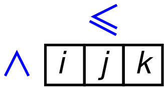
Example:



$s_3(x_1, x_2, \ldots)$

Skew Schur functions

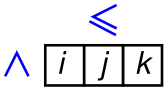
Example:



$$s_3(x_1, x_2, \ldots) = \sum_{i \leq j \leq k} x_i x_j x_k$$

Skew Schur functions

Example:



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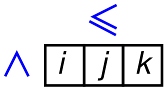
Question: Why do we care about skew Schur functions?

Skew Schur functions

Example:



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- Fact: Skew Schur functions are symmetric in $x_1, x_2, \ldots$.

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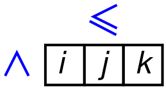

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- Fact: Skew Schur functions are symmetric in $x_1, x_2, \ldots$.
- Fact: The Schur functions form a basis for the algebra of symmetric functions.

Skew Schur functions

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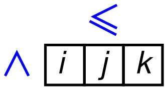

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Question: Why do we care about skew Schur functions?

- Fact: Skew Schur functions are symmetric in $x_1, x_2, \ldots$.
- Fact: The Schur functions form a basis for the algebra of symmetric functions.
- Strong connections with representation theory of S_n and $GL(n, \mathbb{C})$, Schubert Calculus, eigenvalues of Hermitian matrices, ...

The Pieri rule

The (classical) Pieri rule expands $s_\lambda s_n$ in terms of $\{s_\mu\}$.

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Theorem [Pieri, 1893]: For a partition λ and positive integer n,

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where the sum is over all λ^+ such that λ^+/λ is a horizontal strip with n boxes.

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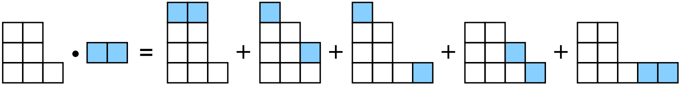
Example:

$$s_{322} s_2 = s_{3222} + s_{3321} + s_{4221} + s_{432} + s_{522}.$$



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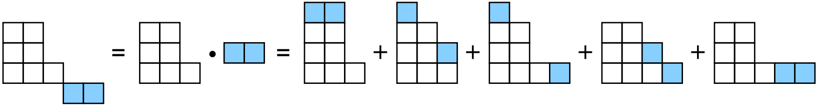

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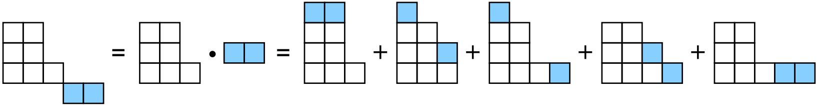

Pieri-type rules in other settings

- k-Schur functions [Lapointe–Morse]
- Schubert polynomials [Lascoux–Schützenberger, Lenart–Sottile, Manivel, Sottile, Winkel]
- LLT polynomials [Lam]
- Schubert classes in the affine Grassmannian [Lam–Lapointe–Morse–Shimozono]
- Hall-Littlewood polynomials [Morris]
- Jack polynomials [Lassalle, Stanley]
- Macdonald polynomials [Koornwinder, Macdonald]
- Quasisymmetric Schur functions [Haglund, Luoto, Mason, van Willigenburg]
- Grothendieck polynomials [Lenart–Sottile]
- Factorial Grothendieck polynomials [McNamara (no relation!)]
- ....

Notably absent: skew Schur functions

The skew Pieri rule

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Theorem [Assaf–McN.]: For a skew shape λ/μ and positive integer n,

$$s_{\lambda/\mu}s_n = \sum_{k=0}^{n}(-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} \ ,$$

where λ^+/λ is a horizontal strip with $n-k$ boxes and μ/μ^- is a vertical strip with k boxes.

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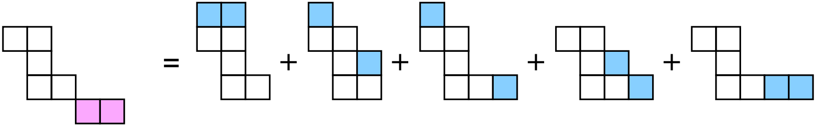

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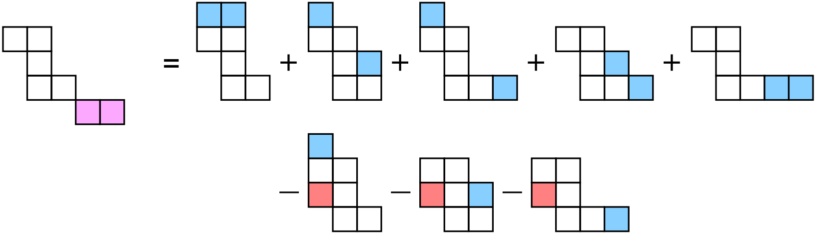

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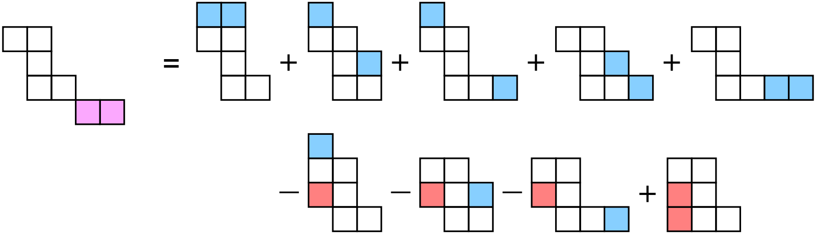
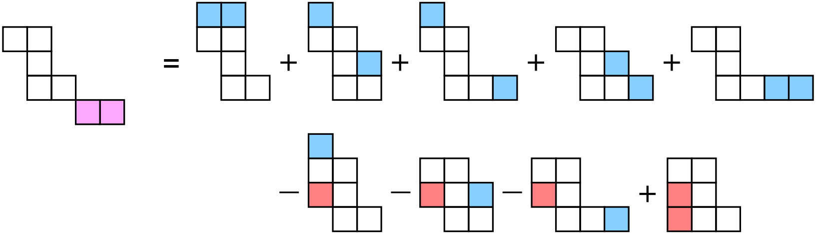


The combinatorial proof

$$s_{\lambda/\mu} s_n = \sum_{k=0}^{n} (-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} \ ,$$



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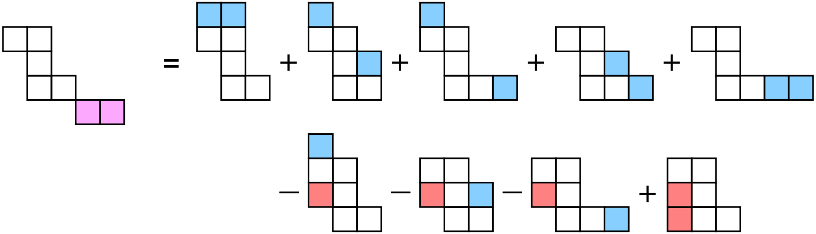
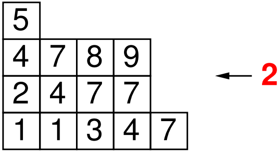


Technique: a sign-reversing involution on SSYT that:

- Preserves entries appearing in each SSYT;
- Has fixed points with $k = 0$ in bijection with SSYT of shape $(\lambda/\mu) * (n)$;
- (Remaining SSYT with k even) $\longleftrightarrow$ (SSYT with k odd).

Robinson-Schensted-Knuth insertion

Robinson-Schensted-Knuth insertion



Robinson-Schensted-Knuth insertion

5				
4	7	8	9	
2	4	7	7	
1	1	3	4	7

$\leftarrow$ **2**

$=$

5					
4	7	8	9		
2	4	7	7		
1	1	3	4	7	$\leftarrow$ **2**

Robinson-Schensted-Knuth insertion

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-4} 4 & 7 & 8 & 9 \\ \cline{1-4} 2 & 4 & 7 & 7 \\ \hline 1 & 1 & \color{red}{\mathbf{3}} & 4 & 7 \\ \hline \end{array} \quad \leftarrow \color{red}{\mathbf{2}} \qquad = \qquad \begin{array}{|c|c|c|c|c|l}\cline{1-1} 5 \\ \cline{1-4} 4 & 7 & 8 & 9 \\ \cline{1-4} 2 & 4 & 7 & 7 \\ \cline{1-5} 1 & 1 & \color{red}{\mathbf{3}} & 4 & 7 & \leftarrow \color{blue}{\mathbf{2}} \\ \cline{1-5} \end{array}$$

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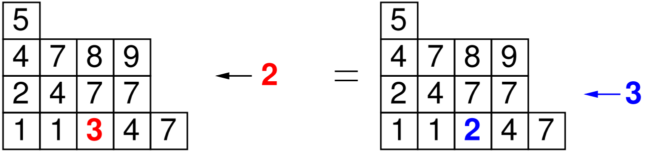

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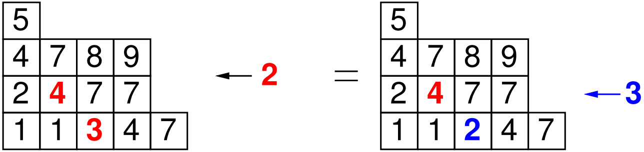

Robinson-Schensted-Knuth insertion

$$
\begin{array}{|c|c|c|c|c|}
\hline
5 \\
\cline{1-1}
4 & 7 & 8 & 9 \\
\cline{1-4}
2 & \mathbf{4} & 7 & 7 \\
\hline
1 & 1 & \mathbf{3} & 4 & 7 \\
\hline
\end{array}
\;\longleftarrow \mathbf{2}
\quad = \quad
\begin{array}{|c|c|c|c|c|}
\hline
5 \\
\cline{1-1}
4 & 7 & 8 & 9 \\
\cline{1-4}
2 & \mathbf{3} & 7 & 7 \\
\hline
1 & 1 & \mathbf{2} & 4 & 7 \\
\hline
\end{array}
\quad \longleftarrow \mathbf{4}
$$

Robinson-Schensted-Knuth insertion

5				
4	7	8	9	
2	4	7	7	
1	1	3	4	7

$\leftarrow$ **2**

$=$

5				
4	7	8	9	
2	3	7	7	
1	1	2	4	7

$\leftarrow$ **4**

Robinson-Schensted-Knuth insertion

$$
\begin{array}{|c|c|c|c|c}
\hline
5 & \multicolumn{4}{c}{} \\
\cline{1-4}
4 & \mathbf{7} & 8 & 9 & \multicolumn{1}{c}{} \\
\cline{1-4}
2 & \mathbf{4} & 7 & 7 & \multicolumn{1}{c}{} \\
\hline
1 & 1 & \mathbf{3} & 4 & \multicolumn{1}{c|}{7} \\
\hline
\end{array}
\;\leftarrow \mathbf{2}
\quad = \quad
\begin{array}{|c|c|c|c|c}
\hline
5 & \multicolumn{4}{c}{} \\
\cline{1-4}
4 & \mathbf{4} & 8 & 9 & \multicolumn{1}{c}{} \\
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\hline
1 & 1 & \mathbf{2} & 4 & \multicolumn{1}{c|}{7} \\
\hline
\end{array}
\qquad \leftarrow \mathbf{7}
$$

Robinson-Schensted-Knuth insertion

$$\begin{array}{|c|c|c|c|c|}\hline 5 \\ \cline{1-1} 4 & 7 & 8 & 9 \\ \hline 2 & 4 & 7 & 7 \\ \hline 1 & 1 & 3 & 4 & 7 \\ \hline \end{array} \quad \leftarrow 2 \quad = \quad \begin{array}{|c|c|c|c|c|}\hline 5 & 7 \\ \hline 4 & 4 & 8 & 9 \\ \hline 2 & 3 & 7 & 7 \\ \hline 1 & 1 & 2 & 4 & 7 \\ \hline \end{array}$$

Robinson-Schensted-Knuth insertion

5				
4	**7**	8	9	
2	**4**	7	7	
1	1	**3**	4	7

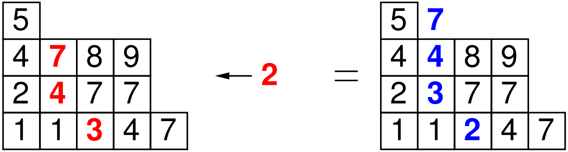
$\leftarrow$ **2** $=$

5	**7**			
4	**4**	8	9	
2	**3**	7	7	
1	1	**2**	4	7

Facts:

- The result is an SSYT.
- This process is reversible.

Robinson-Schensted-Knuth insertion



Facts:

- The result is an SSYT.
- This process is reversible.
- RSK insertion can be used to give a combinatorial proof of the classicial Pieri rule.

External insertion (just like RSK):

Sagan & Stanley's generalization to skew shapes

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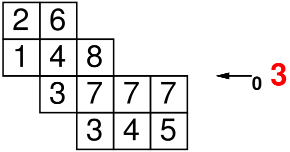

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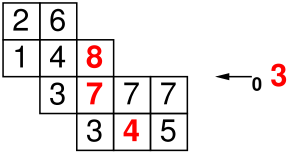

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External insertion (just like RSK):

$$\begin{array}{ccccc} 2 & 6 & & & \\ 1 & 4 & \mathbf{8} & & \\ & 3 & \mathbf{7} & 7 & 7 \\ & & 3 & \mathbf{4} & 5 \end{array} \xleftarrow{\ }_{0} \mathbf{3} = \begin{array}{ccccc} 2 & 6 & \mathbf{8} & & \\ 1 & 4 & \mathbf{7} & & \\ & 3 & \mathbf{4} & 7 & 7 \\ & & 3 & \mathbf{3} & 5 \end{array}$$

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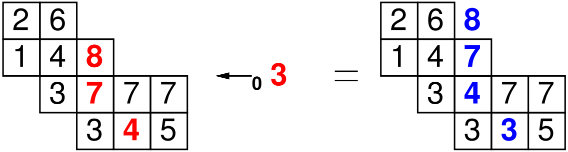
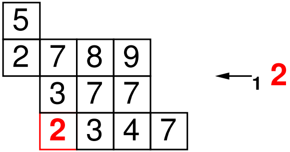


Internal insertion:



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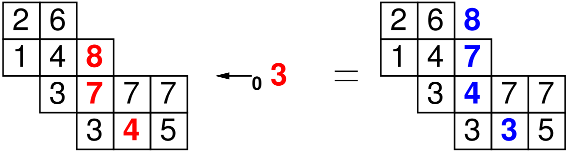

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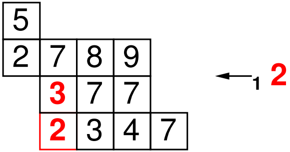

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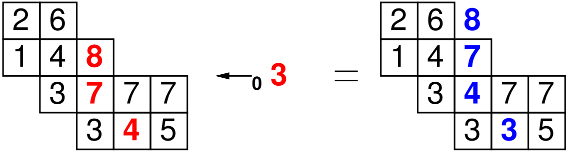

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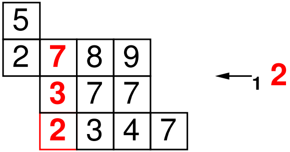

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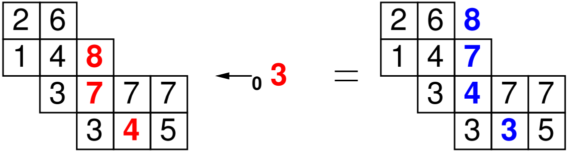

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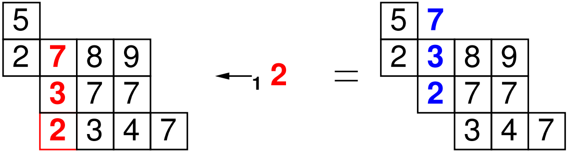
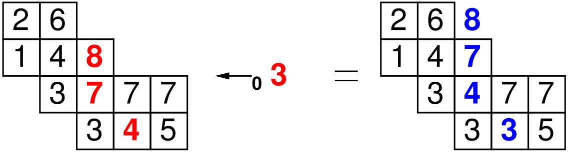
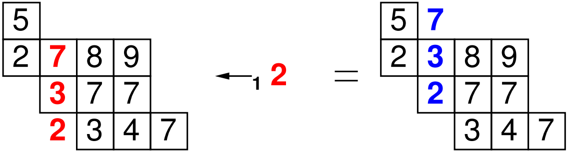


Sagan & Stanley's generalization to skew shapes

External insertion (just like RSK):



Internal insertion:



Sagan & Stanley's generalization to skew shapes

External insertion (just like RSK):

2	6			
1	4	**8**		
	3	**7**	7	7
		3	**4**	5

$\leftarrow_0$ **3** $=$

2	6	**8**		
1	4	**7**		
	3	**4**	7	7
		3	**3**	5

Internal insertion:

5				
2	**7**	8	9	
	3	7	7	
	2	3	4	7

$\leftarrow_1$ **2** $=$

5	**7**			
2	**3**	8	9	
	2	7	7	
		3	4	7

Would be great: if internal insertion gave the necessary bijection for skew Pieri rule.

Sagan & Stanley's generalization to skew shapes

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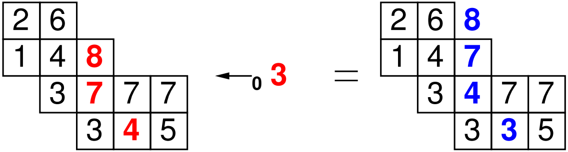

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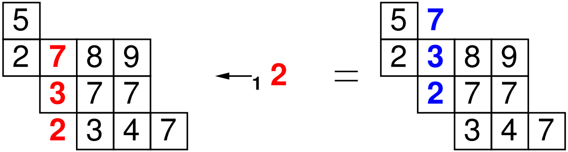
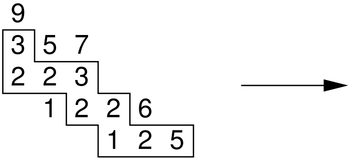


Would be great: if internal insertion gave the necessary bijection for skew Pieri rule.

In general, it's not that easy....

The sign-reversing involution on SSYT of form λ^+/μ^-

Example 1: reverse insert until you perform a reverse internal insertion. Then externally insert the overflow.



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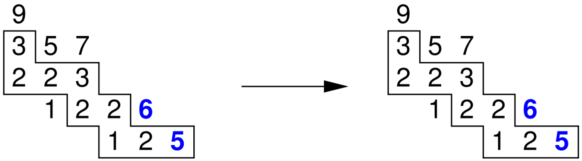

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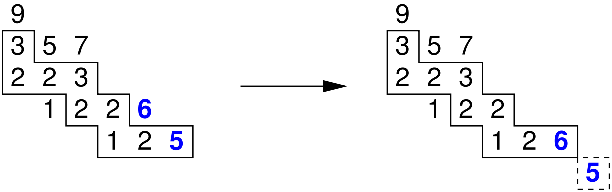

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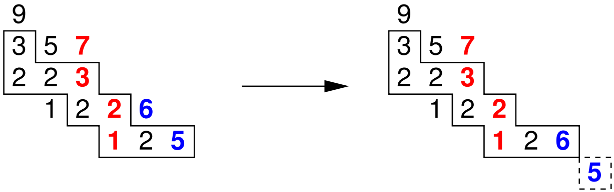

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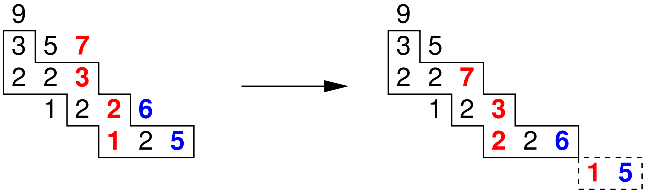

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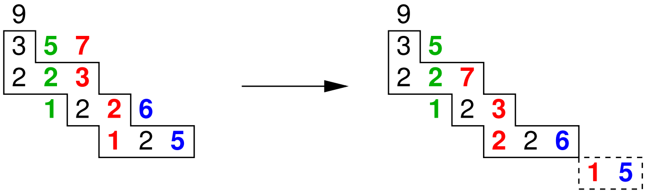

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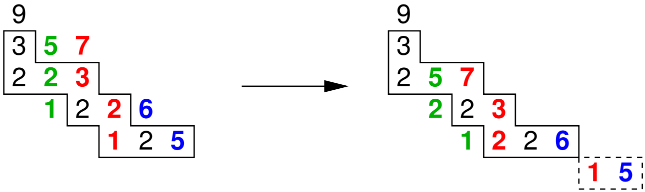

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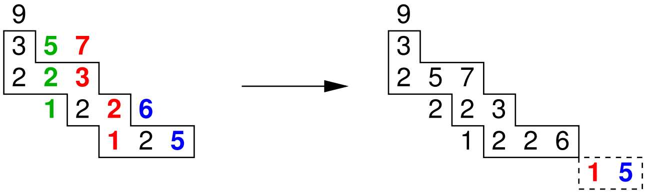

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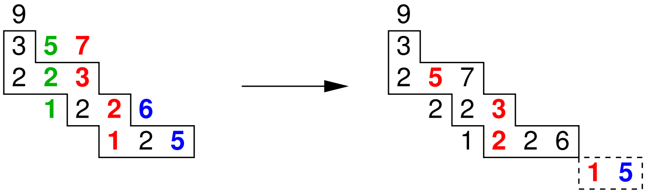

The sign-reversing involution on SSYT of form λ^+/μ^-

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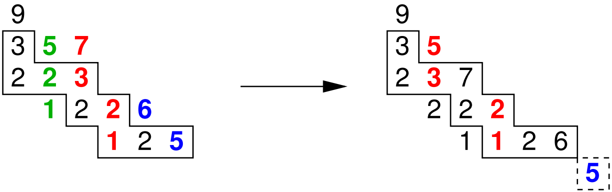
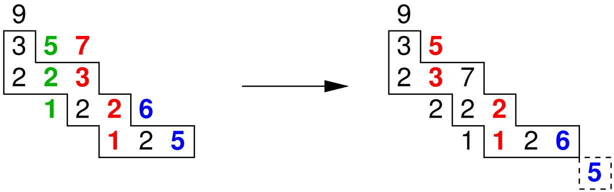


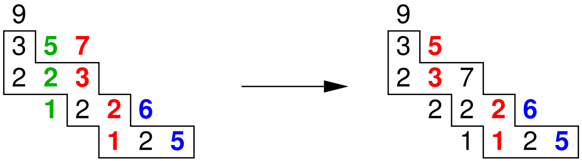
The sign-reversing involution on SSYT of form λ^+/μ^-

Example 1: reverse insert until you perform a reverse internal insertion. Then externally insert the overflow.



The sign-reversing involution on SSYT of form λ^+/μ^-

Example 1: reverse insert until you perform a reverse internal insertion. Then externally insert the overflow.



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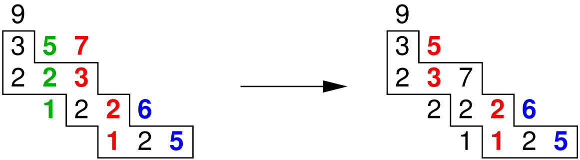
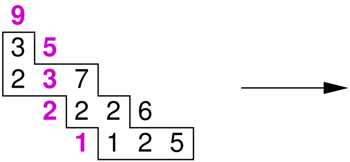


Example 2: but stop if you're left of an upward path. Then perform the internal insertion, and then insert the overflow.



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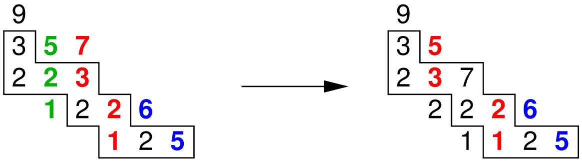

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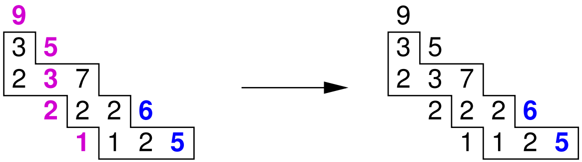

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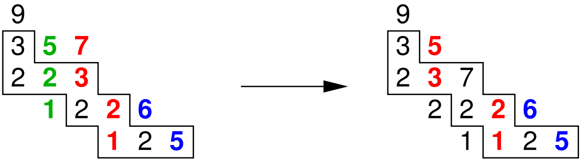
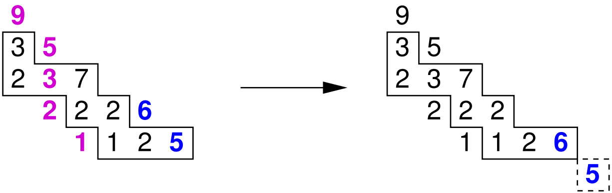
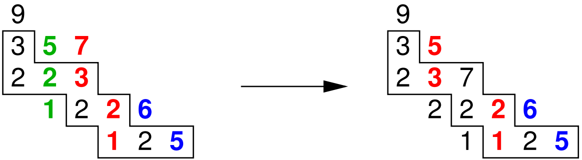


Example 2: but stop if you're left of an upward path. Then perform the internal insertion, and then insert the overflow.



The sign-reversing involution on SSYT of form λ^+/μ^-

Example 1: reverse insert until you perform a reverse internal insertion. Then externally insert the overflow.



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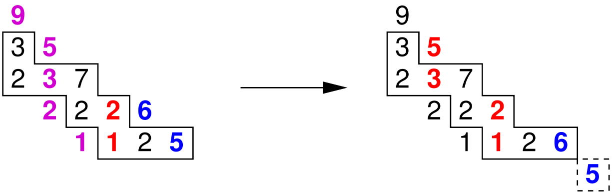

The sign-reversing involution on SSYT of form λ^+/μ^-

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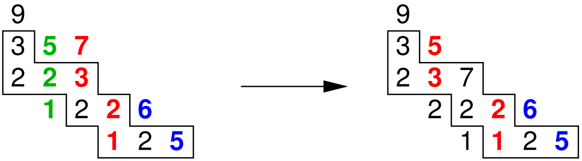

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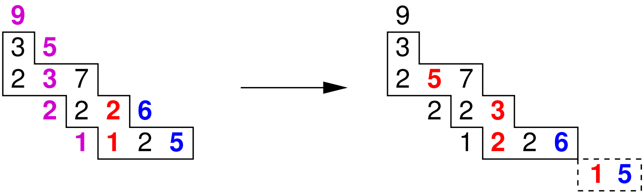

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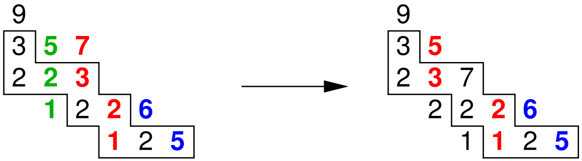

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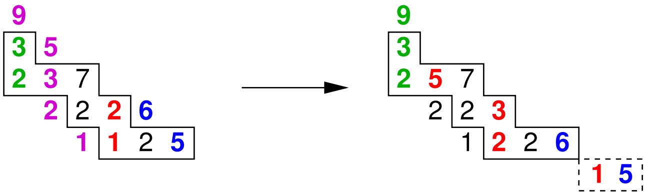

The sign-reversing involution on SSYT of form λ^+/μ^-

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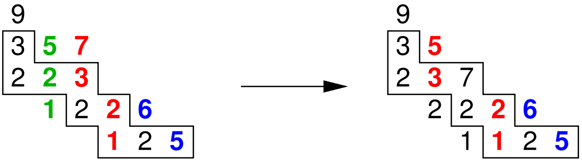

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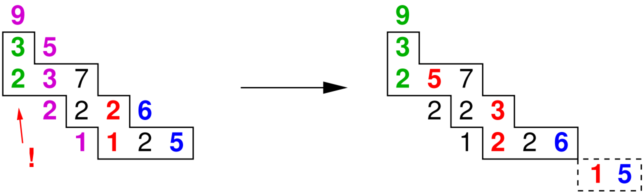

The sign-reversing involution on SSYT of form λ^+/μ^-

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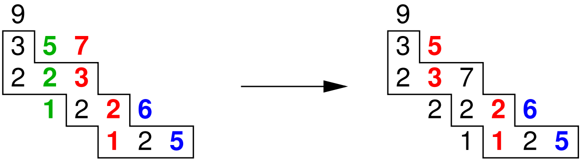

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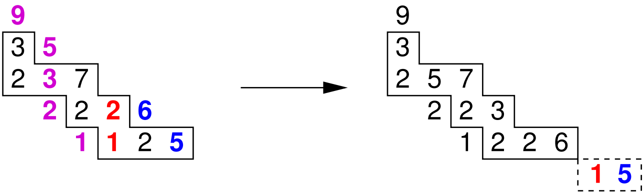

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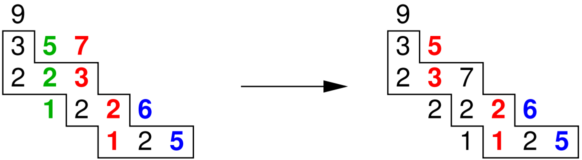

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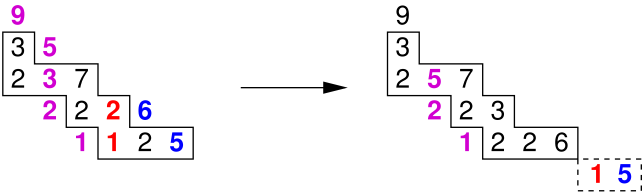

The sign-reversing involution on SSYT of form λ^+/μ^-

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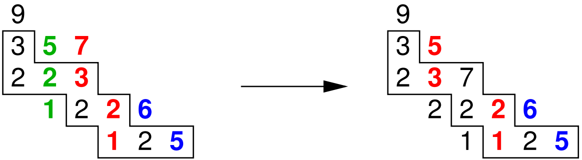

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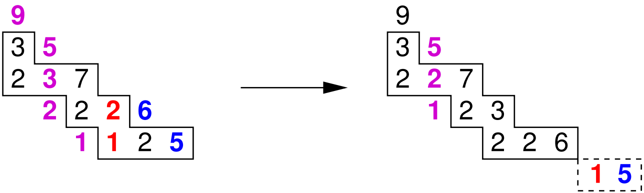

The sign-reversing involution on SSYT of form λ^+/μ^-

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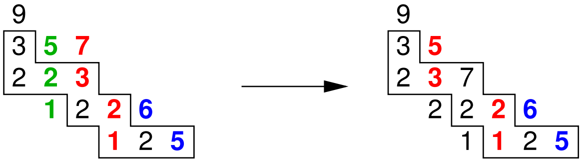

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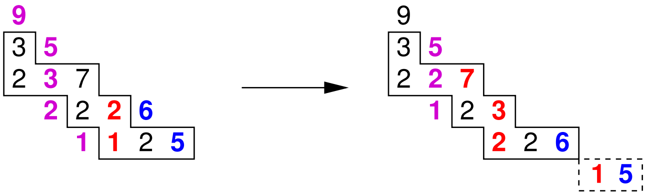

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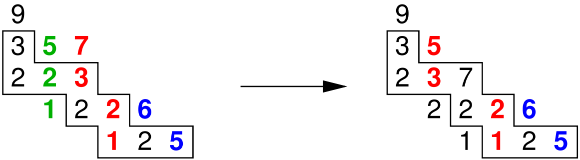

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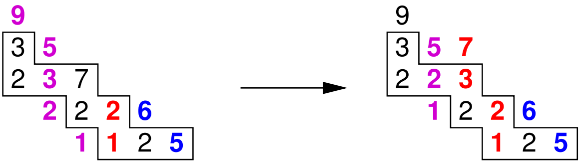

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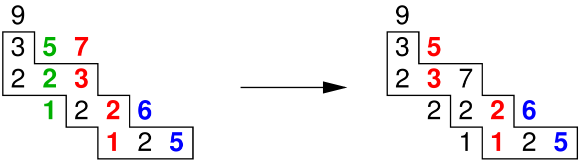

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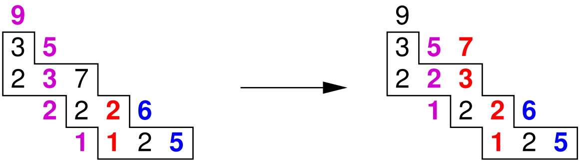
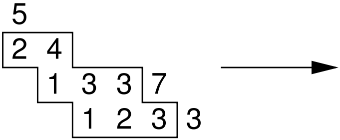


Bijection between these two types that is sign-reversing.

Example 3: Fixed points.

These should be in bijection with SSYT of shape $(\lambda/\mu) * (n)$.



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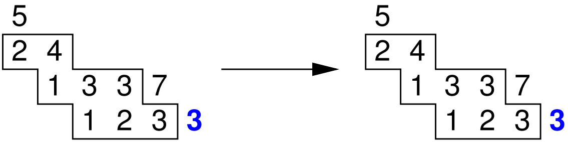

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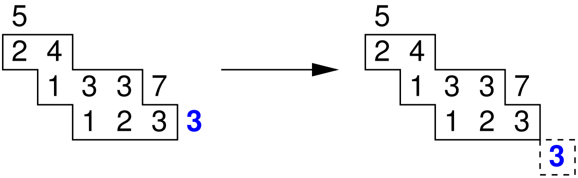

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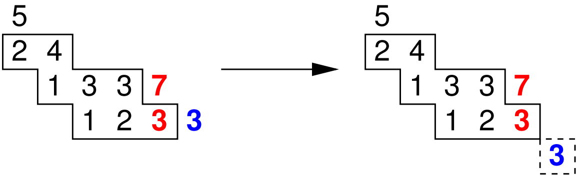

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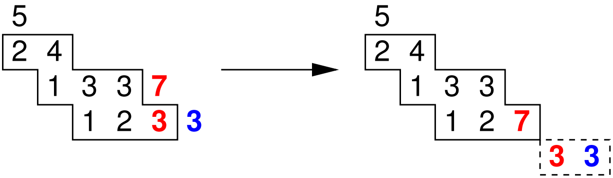

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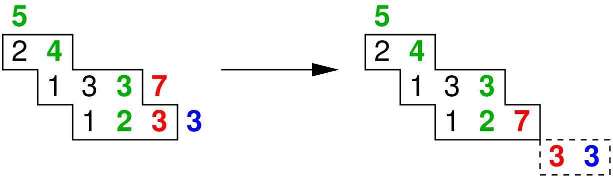

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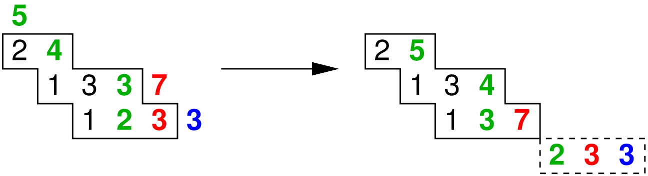

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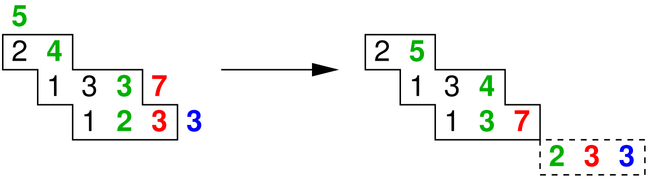

Conclusion:

$$s_{\lambda/\mu} s_n = \sum_{k=0}^{n} (-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} \ ,$$

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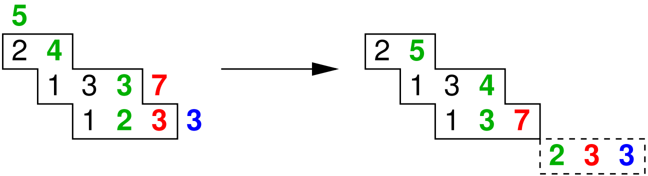

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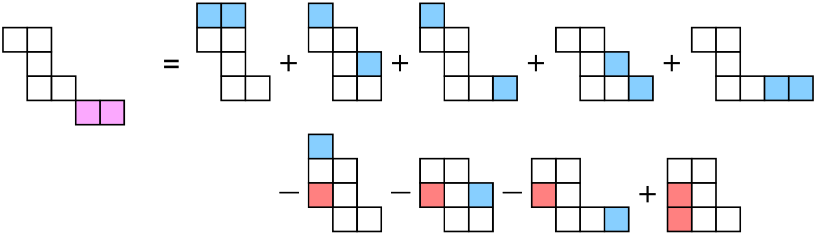
Proof 2: Algebraic proof given by Thomas Lam (as appendix in full paper).

Development 1: Rectification one row at a time

A possible application of the skew Pieri rule:



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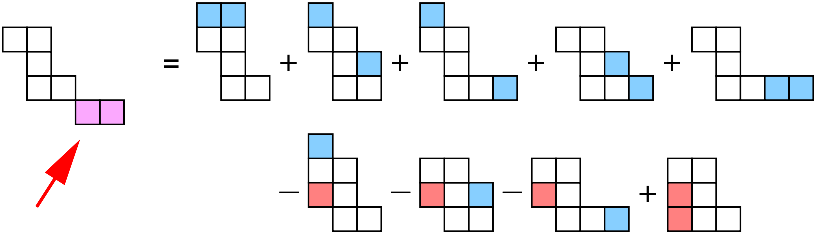

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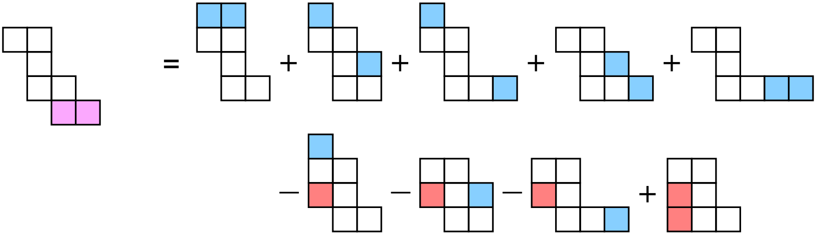

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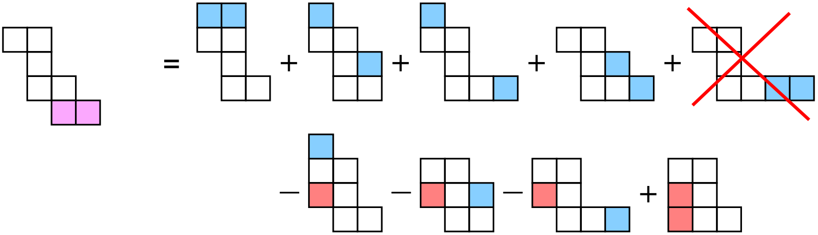

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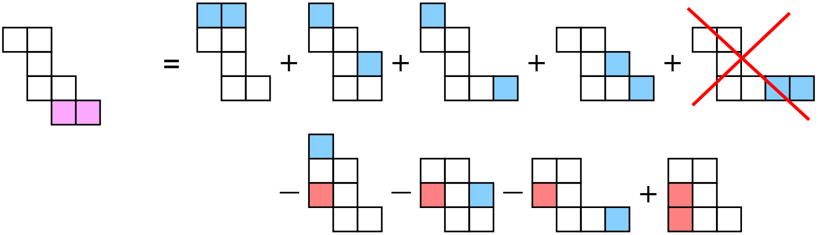

Exactly the same proof works.

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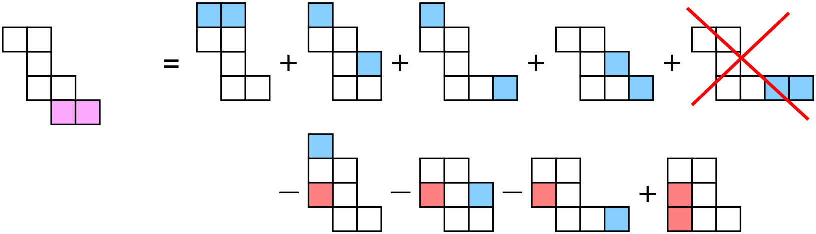


Exactly the same proof works.

Allows you to rectify a skew shape (i.e. expand $s_{\lambda/\mu}$ in terms of $\{s_\nu\}$) one row at a time.

Development 2: Skew Littlewood–Richardson rule

Conjecture [Assaf, McN.]: An expansion of $s_{\lambda/\mu}s_{\sigma/\tau}$ in terms of $\{s_{\lambda^+/\mu^-}\}$ that generalizes the skew Pieri rule.

Exact statement is coming up in Aaron's talk (in terms of jeu-de-taquin).

Proof [Lam–Lauve–Sottile]: using Hopf algebras.

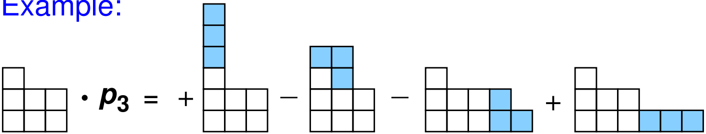
Open problem: find a combinatorial proof.

Theorem: For λ and a positive integer n,

$$s_\lambda \quad p_n = \sum_{\lambda^+} (-1)^{ht(\lambda^+/\lambda)} s_{\lambda^+}$$

where λ^+/λ is a ribbon with n boxes

Example:



Development 3: 2 combinatorial proofs for the \$ of 1

Theorem: For λ/μ and a positive integer n,

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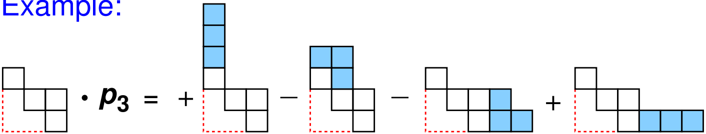
Example:

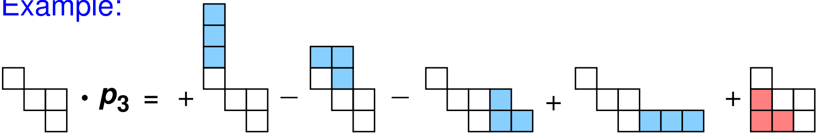


Theorem: For λ/μ and a positive integer n,

$$s_{\lambda/\mu} p_n = \sum_{\lambda^+} (-1)^{ht(\lambda^+/\lambda)} s_{\lambda^+/\mu} - \sum_{\mu^-} (-1)^{ht(\mu/\mu^-)} s_{\lambda/\mu^-}$$

where λ^+/λ is a ribbon with n boxes and so is μ/μ^-.

Example:



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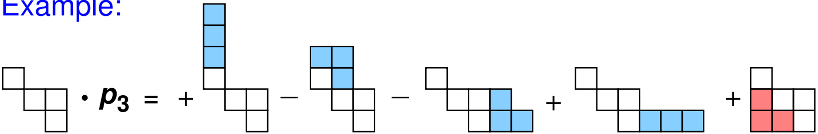

Question for you: is this a new result?

Proof 1: Algebraic. Special case of LLS Hopf Formula Lemma (or à la Lam's skew Pieri proof, but easier).

Proof 2 [McN.?]: Combinatorial, except that it uses skew Littlewood-Richardson rule.

Development 3: 2 combinatorial proofs for the $ of 1

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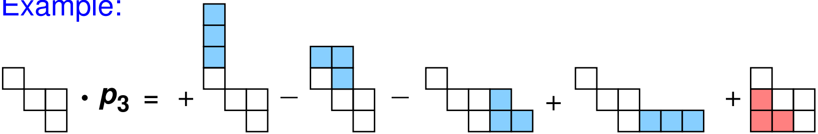
Example:



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Proof 1: Algebraic. Special case of LLS Hopf Formula Lemma (or à la Lam's skew Pieri proof, but easier).

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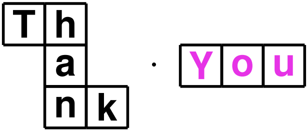
Easier(?) open problem: or find a combinatorial proof that doesn't need the skew LR-rule.

Full paper available on the arXiv:

Sami H. Assaf and Peter R.W. McNamara. *A Pieri rule for skew shapes*, JCT-A, to appear, arXiv:0908.0345



The End

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