A Combinatorial Classification of Skew Schur Functions

Peter McNamara
Bucknell University

Joint work with Stephanie van Willigenburg

FPSAC 2007
Nankai University, Tianjin, China
3 July 2007

Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

When are Two Skew Schur Functions Equal?

Peter McNamara
Bucknell University

Joint work with Stephanie van Willigenburg

FPSAC 2007
Nankai University, Tianjin, China
3 July 2007

Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

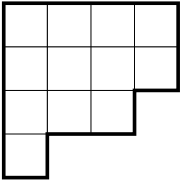
Outline

- Background: skew Schur functions
- Recent work on skew Schur function equality
- Composition of skew diagrams, main results
- Conjectures, open problems

Schur functions

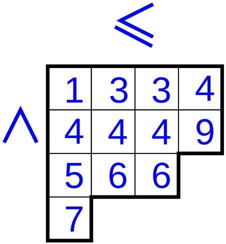
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- Young diagram.
  Example:
  $\lambda = (4, 4, 3, 1)$
- Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

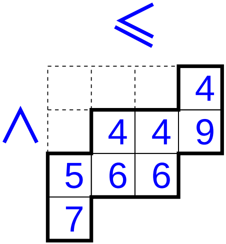
$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431/31} = \quad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

Notes on skew Schur functions

- Skew Schur functions are symmetric in the variables $x = (x_1, x_2, \ldots)$.
- The Schur functions form a basis for the algebra of symmetric functions (over $\mathbb{Q}$, say).
- Connections with Algebraic Geometry, Representation Theory.

The HDL series

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

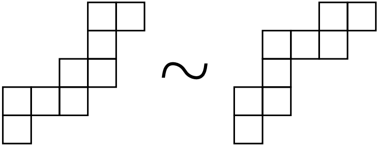
- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

The HDL series

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

Complete classification of equality of ribbon Schur functions



- HDL II: Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006):
  - The more general setting of binomial syzygies

    $$c\, s_{D_1} s_{D_2} \cdots s_{D_m} = c'\, s_{E_1} s_{E_2} \cdots s_{E_n}$$

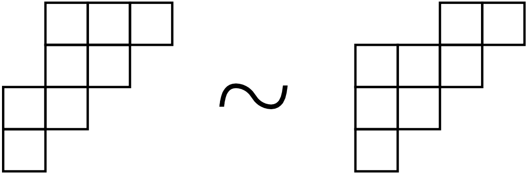
    is equivalent to understanding equalities among connected skew diagrams.
  - 3 operations for generating skew diagrams with equal skew Schur functions.
  - For $\#boxes \leq 18$, there are 6 examples that escape explanation.
  - Necessary conditions, but of a different flavor.

The HDL series

- HDL III: McN., Steph van Willigenburg (2006):
  - An operation that encompasses the operation of HDL I and the three operations of HDL II.
  - Theorem that generalizes all previous results. Explains all equivalences where $\#boxes \leq 20$.
  - Conjecture for necessary and sufficient conditions for $s_{\lambda/\alpha} = s_{\mu/\beta}$. Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes) D, E.
If $s_D = s_E$, we will write $D \sim E$.

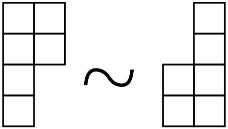
Example



We want to classify all equivalences classes, thereby classifying all skew Schur functions.

The basic building block

EC2, Exercise 7.56(a) [2-]

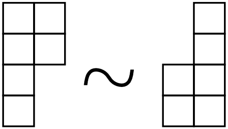


Theorem

$D \sim D^*$, where D^* denotes D rotated by $180°$.

The basic building block

EC2, Exercise 7.56(a) [2-]



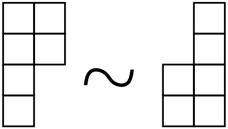
Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

Goal: Use this equivalence to build other skew equivalences.

The basic building block

EC2, Exercise 7.56(a) [2-]



Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

Goal: Use this equivalence to build other skew equivalences.

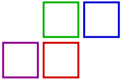
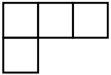
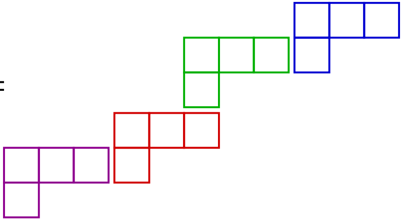
Where we're headed:

Theorem

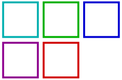
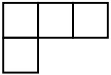
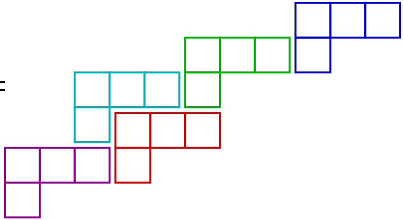
Suppose we have skew diagrams D, D' and E satisfying certain assumptions. If $D \sim D'$ then

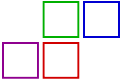
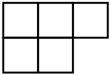
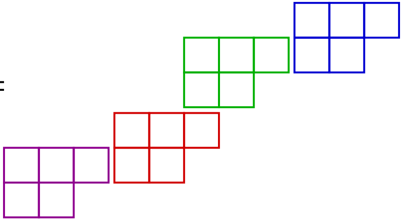
$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$

Main definition: composition of skew diagrams.

$D \circ E =$  $\circ$  $=$ 

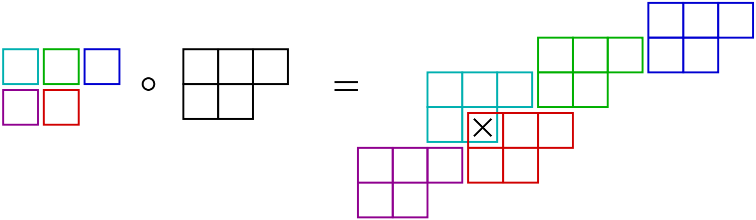
Composition of skew diagrams

$D \circ E =$  $\circ$  $=$ 

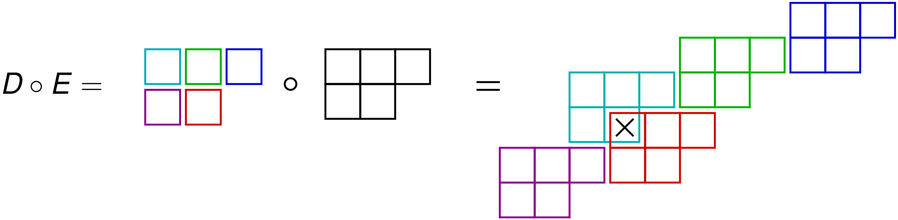
$D \circ E =$  $\circ$  $=$ 

Composition of skew diagrams

$D \circ E =$

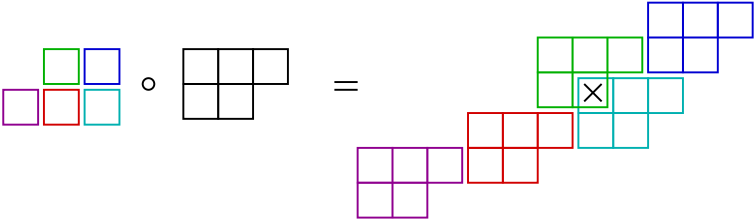


Composition of skew diagrams



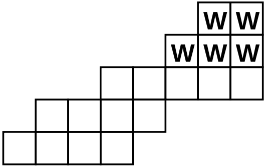
Theorem [McN., van Willigenburg] *If $D \sim D'$, then*

$$D' \circ E \sim D \circ E \sim D \circ E^*.$$



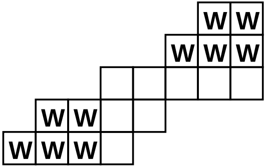
Amalgamated Compositions

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



Amalgamated Compositions

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.

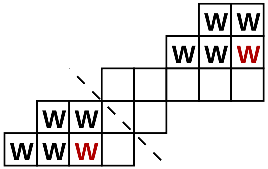


Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

Amalgamated Compositions

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



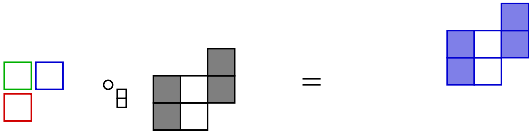
Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

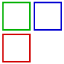
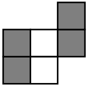
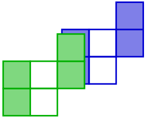
Hypotheses: (inspired by hypotheses of RSvW)

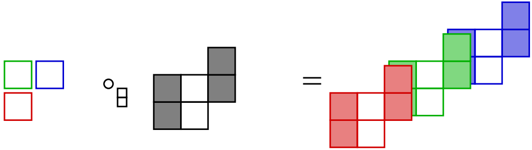
1. W_{ne} and W_{sw} are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew diagrams.
3. W is maximal given its set of diagonals.

Amalgamated Compositions

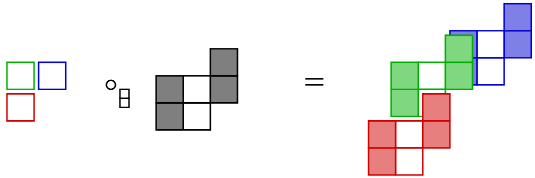
$D \circ_W E =$ 

Amalgamated Compositions

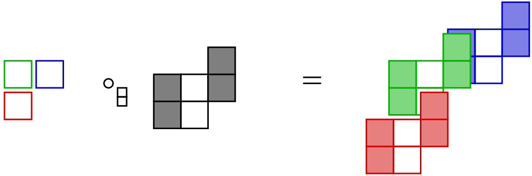
$D \circ_W E =$  $\circ$  $=$ 

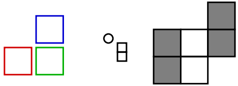
$D \circ_W E =$ 

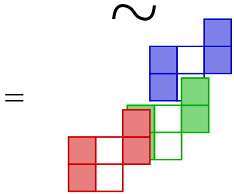
$D \circ_W E =$



$D \circ_W E =$

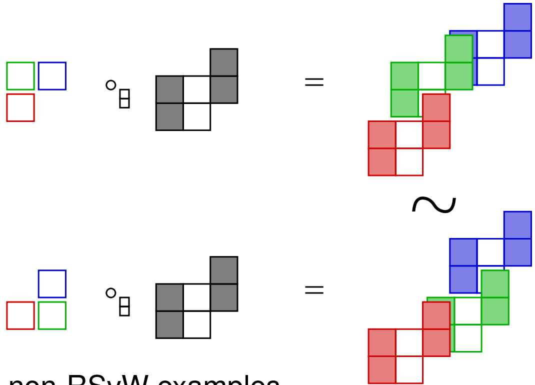






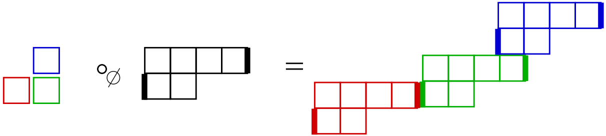
15 boxes: first of the non-RSvW examples

$D \circ_W E =$



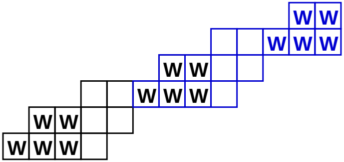
15 boxes: first of the non-RSvW examples

If $W = \emptyset$, we get the regular compositions:



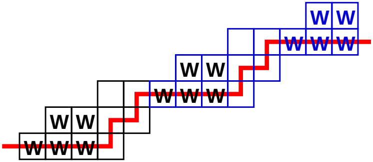
What are the results?

Construction of $\overline{W}$ and $\overline{O}$:



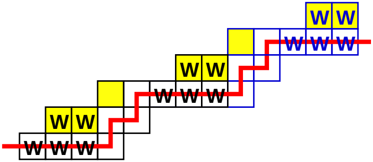
What are the results?

Construction of $\overline{W}$ and $\overline{O}$:



What are the results?

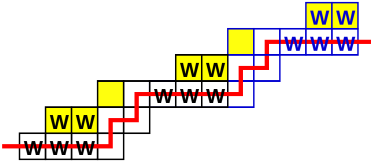
Construction of $\overline{W}$ and $\overline{O}$:



Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

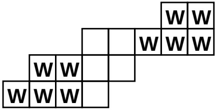
What are the results?

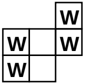
Construction of $\overline{W}$ and $\overline{O}$:



Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

Hypothesis 5. In $E = WOW$, at least one copy of W has just one cell adjacent to O.





What are the results?

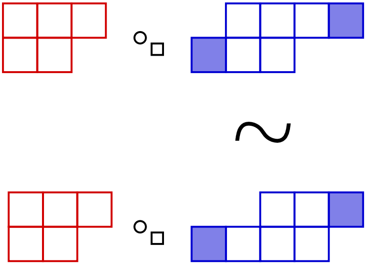
Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5. Then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$

What are the results?

Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5. Then

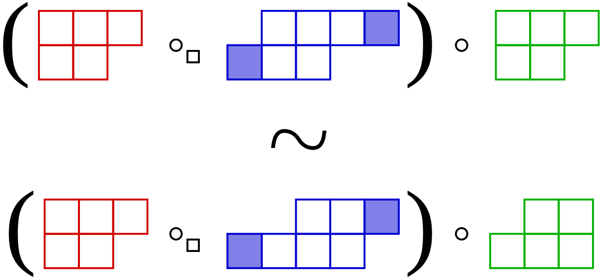
$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$



What are the results?

Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5. Then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$



is a skew equivalence with 145 boxes.

A word about the proof

The key: An expression for $s_{D \circ_W E}$ in terms of s_D, s_E, $s_{\overline{W}}$, $s_{\overline{O}}$.

Proof of expression uses:

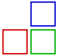
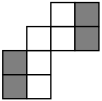
- Hamel-Goulden determinants.
  Angèle M. Hamel and Ian P. Goulden:
  Planar decompositions of tableaux and Schur function determinants.

  William Y. C. Chen, Guo-Guang Yan and Arthur L. B. Yang:
  Transformations of border strips and Schur function determinants.

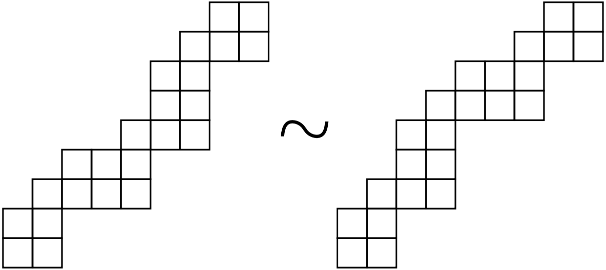
- Sylvester's Determinantal Identity.

Open problems

- Removing Hypothesis 5 (at least one copy of W has just one cell adjacent to O).

$D =$  $E =$ 

$D \circ_W E$ has 23 boxes, and $D \circ_W E \sim D^* \circ_W E$:



(Software of Anders Buch, John Stembridge)

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \ \sim \ ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \text{ where} \end{aligned}$$

- $\circ$ $E_i = W_i O_i W_i$ satisfies Hypotheses 1-4 for all i,
- $\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \end{aligned}$$ where

- $\circ$ $E_i = W_i O_i W_i$ satisfies Hypotheses 1-4 for all i,
- $\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

True when $\#boxes \leq 20$.

Summary

- A definition of skew diagram composition. Encompasses the operation of BTvW and the three operations of RSvW.
- Theorem that generalizes all previous results.
  In particular, explains the 6 missing equivalences from RSvW.
- Conjecture for necessary and sufficient conditions for $E \sim E'$.