

A Combinatorial Classification of Skew Schur Functions

Peter McNamara
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Joint work with Stephanie van Willigenburg

FPSAC 2007
Nankai University, Tianjin, China
3 July 2007

Slides and paper available from
www.facstaff.bucknell.edu/pm040/

When are Two Skew Schur Functions Equal?

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- ▶ Background: skew Schur functions
- ▶ Recent work on skew Schur function equality
- ▶ Composition of skew diagrams, main results
- ▶ Conjectures, open problems

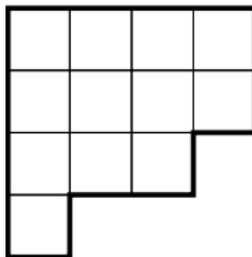
Schur functions

▶ Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

▶ Young diagram.

Example:

$$\lambda = (4, 4, 3, 1)$$



Schur functions

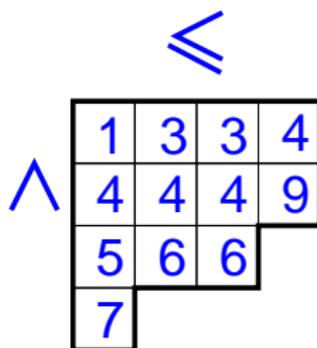
- ▶ Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

- ▶ Young diagram.

Example:

$$\lambda = (4, 4, 3, 1)$$

- ▶ Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \dots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \dots$$

Example

$$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \dots$$

Skew Schur functions

▶ Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

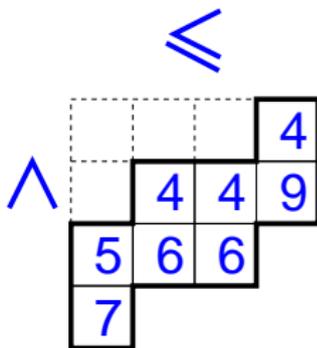
▶ μ fits inside λ .

▶ Young diagram.

Example:

$$\lambda/\mu = (4, 4, 3, 1)/(3, 1)$$

▶ Semistandard Young tableau (SSYT)



The **skew** Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \dots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \dots$$

Example

$$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \dots$$

- ▶ Skew Schur functions are symmetric in the variables $x = (x_1, x_2, \dots)$.
- ▶ The Schur functions form a basis for the algebra of symmetric functions (over \mathbb{Q} , say).
- ▶ Connections with Algebraic Geometry, Representation Theory.

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

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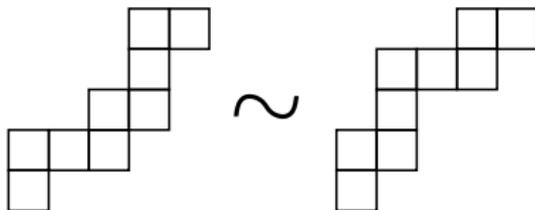


Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

- ▶ Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):



Complete classification of equality of **ribbon** Schur functions



- ▶ HDL II: [Vic Reiner, Kristin Shaw, Steph van Willigenburg \(2006\)](#):
 - ▶ The more general setting of binomial syzygies

$$c s_{D_1} s_{D_2} \cdots s_{D_m} = c' s_{E_1} s_{E_2} \cdots s_{E_n}$$

is equivalent to understanding equalities among connected skew diagrams.

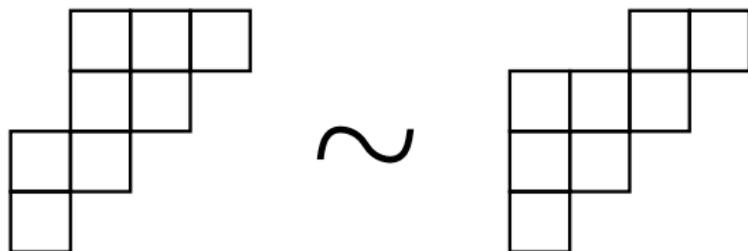
- ▶ 3 operations for generating skew diagrams with equal skew Schur functions.
- ▶ For $\#boxes \leq 18$, there are 6 examples that escape explanation.
- ▶ Necessary conditions, but of a different flavor.

- ▶ HDL III: McN., Steph van Willigenburg (2006):
 - ▶ An operation that encompasses the operation of HDL I and the three operations of HDL II.
 - ▶ Theorem that generalizes all previous results.
Explains all equivalences where $\#boxes \leq 20$.
 - ▶ Conjecture for necessary and sufficient conditions for $s_{\lambda/\alpha} = s_{\mu/\beta}$.
Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes) D, E .

If $s_D = s_E$, we will write $D \sim E$.

Example



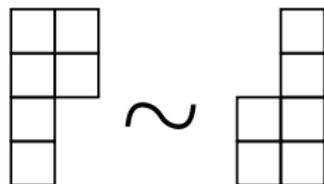
We want to classify all equivalence classes, thereby classifying all skew Schur functions.

The basic building block

EC2, Exercise 7.56(a) [2-]

Theorem

$D \sim D^*$, where D^* denotes D rotated by 180° .

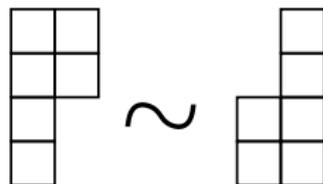


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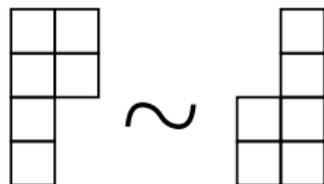
Goal: Use this equivalence to build other skew equivalences.

The basic building block

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Goal: Use this equivalence to build other skew equivalences.

Where we're headed:

Theorem

Suppose we have skew diagrams D , D' and E satisfying certain assumptions. If $D \sim D'$ then

$$D' \circ E \sim D \circ E \sim D \circ E^*.$$

Main definition: composition of skew diagrams.

Composition of skew diagrams

$$D \circ E =$$

The diagram illustrates the composition of two skew diagrams, D and E , resulting in a new skew diagram $D \circ E$.

Diagram D (left) consists of a 2x2 grid of cells. The top-left cell is purple, the top-right cell is red, the bottom-left cell is green, and the bottom-right cell is blue.

Diagram E (middle) consists of a 2x3 grid of cells. The top row has three black cells, and the bottom row has one black cell on the left.

The resulting diagram $D \circ E$ (right) is a staircase shape composed of four colored segments: a purple segment (bottom-left), a red segment (middle-left), a green segment (top-left), and a blue segment (top-right).

Composition of skew diagrams

$$D \circ E =$$

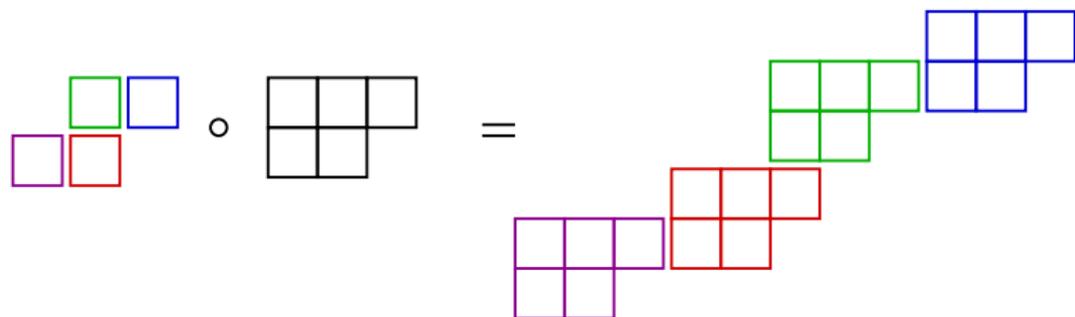
The diagram illustrates the composition of two skew diagrams, D and E , resulting in a larger skew diagram. The composition is shown as $D \circ E =$ followed by the three diagrams.

Diagram D (left) consists of a 2x3 grid of cells. The top row has three cells colored cyan, green, and blue. The bottom row has two cells colored purple and red.

Diagram E (middle) consists of a 2x3 grid of black cells. The top row has three cells, and the bottom row has one cell under the first cell.

The resulting diagram (right) is a 4x6 grid of colored cells. The top row has six cells colored cyan, cyan, cyan, green, green, green. The second row has five cells colored cyan, cyan, cyan, green, blue. The third row has six cells colored purple, purple, purple, red, red, red. The bottom row has five cells colored purple, red, red, red, blue, blue.

Composition of skew diagrams

$$D \circ E =$$


The diagram illustrates the composition of two skew diagrams, D and E , resulting in a larger skew diagram. The composition is shown as $D \circ E =$ followed by the diagrams.

Diagram D (left) consists of a 2x2 grid of squares. The top row has a purple square on the left and a green square on the right. The bottom row has a red square on the left and a blue square on the right.

Diagram E (middle) consists of a 2x3 grid of squares. The top row has three black squares. The bottom row has two black squares, starting from the left.

The result (right) is a 2x6 grid of squares. The top row has six squares: purple, purple, purple, green, green, green. The bottom row has six squares: red, red, red, blue, blue, blue. The bottom row starts from the left, and the top row starts from the left.

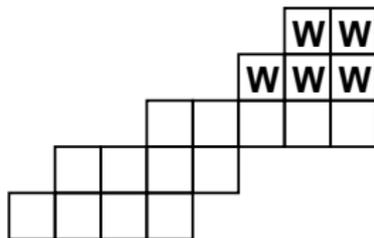
Composition of skew diagrams

$$D \circ E = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array}$$

The diagram on the right shows the composition of the two diagrams. The top row consists of four boxes: purple, cyan, green, and blue. The second row consists of four boxes: purple, cyan, red, and blue. The third row consists of two boxes: purple and red. The fourth row consists of two boxes: purple and red. The fifth row consists of two boxes: purple and red. The sixth row consists of two boxes: purple and red. The seventh row consists of two boxes: purple and red. The eighth row consists of two boxes: purple and red. The ninth row consists of two boxes: purple and red. The tenth row consists of two boxes: purple and red. The eleventh row consists of two boxes: purple and red. The twelfth row consists of two boxes: purple and red. The thirteenth row consists of two boxes: purple and red. The fourteenth row consists of two boxes: purple and red. The fifteenth row consists of two boxes: purple and red. The sixteenth row consists of two boxes: purple and red. The seventeenth row consists of two boxes: purple and red. The eighteenth row consists of two boxes: purple and red. The nineteenth row consists of two boxes: purple and red. The twentieth row consists of two boxes: purple and red. The twenty-first row consists of two boxes: purple and red. The twenty-second row consists of two boxes: purple and red. The twenty-third row consists of two boxes: purple and red. The twenty-fourth row consists of two boxes: purple and red. The twenty-fifth row consists of two boxes: purple and red. The twenty-sixth row consists of two boxes: purple and red. The twenty-seventh row consists of two boxes: purple and red. The twenty-eighth row consists of two boxes: purple and red. The twenty-ninth row consists of two boxes: purple and red. The thirtieth row consists of two boxes: purple and red. The thirtieth row contains a black 'X' in the cyan box.

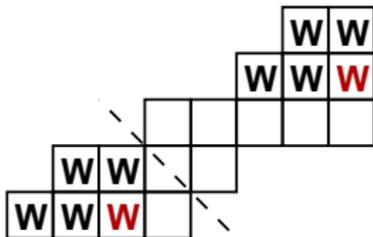
Amalgamated Compositions

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E .



Amalgamated Compositions

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E .



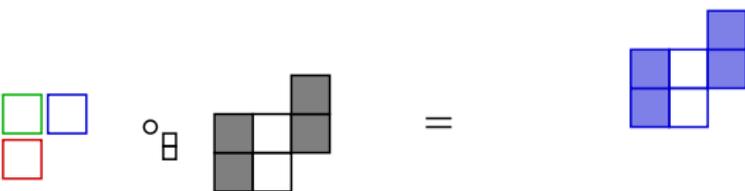
Similarly, W *lies in the bottom* of E .

Our interest: W lies in both the top and bottom of E . We write $E = WOW$.

Hypotheses: (inspired by hypotheses of RSvW)

1. W_{ne} and W_{sw} are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew diagrams.
3. W is maximal given its set of diagonals.

Amalgamated Compositions

$$D \circ_W E =$$


The diagram illustrates the amalgamated composition $D \circ_W E$. On the left, the composition D consists of three boxes: a green box, a blue box, and a red box. The composition E consists of six boxes: a grey box, a white box, a grey box, a grey box, a white box, and a grey box. The result of the amalgamated composition is a composition of six boxes: a blue box, a white box, a blue box, a white box, a blue box, and a blue box.

Amalgamated Compositions

$$D \circ_W E =$$

The diagram illustrates the amalgamated composition $D \circ_W E$. On the left, composition D consists of three boxes: a green box, a blue box, and a red box. Composition E consists of six boxes: a gray box, a white box, a gray box, a white box, a gray box, and a white box. The result is a composition of nine boxes where the green and blue boxes from D are interleaved with the gray and white boxes from E .

Amalgamated Compositions

$$D \circ_W E =$$

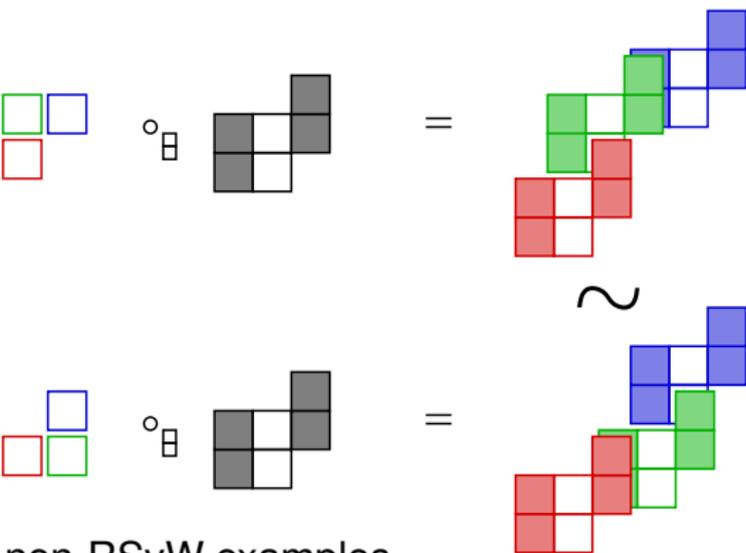
The diagram illustrates the operation $D \circ_W E$. On the left, composition E is shown as a sequence of three boxes: a green box, a blue box, and a red box. Composition D is shown as a sequence of four boxes: a grey box, a white box, a grey box, and a white box. A small circle with a vertical bar below it is positioned between E and D . An equals sign follows, leading to the result of the operation. The result is a sequence of boxes where the red box from E is placed in front of the grey and white boxes of D , and the green and blue boxes from E are placed in front of the remaining grey and white boxes of D . The final sequence consists of: a red box, a white box, a red box, a green box, a white box, a green box, a blue box, a white box, and a blue box.

Amalgamated Compositions

$$D \circ_W E =$$

The diagram illustrates the amalgamated composition $D \circ_W E$. On the left, composition D consists of three boxes: a green box, a blue box, and a red box. Composition E consists of six boxes: a gray box, a white box, a gray box, a white box, a gray box, and a white box. The composition operation \circ_W is shown between them. The result is a composition of nine boxes: three red boxes, three green boxes, and three blue boxes, arranged in a staircase pattern.

Amalgamated Compositions

$$D \circ_W E =$$


15 boxes: first of the non-RSvW examples

Amalgamated Compositions

$$D \circ_W E = \begin{array}{c} \square \square \\ \square \end{array} \circ_{\begin{array}{c} \square \\ \square \end{array}} \begin{array}{c} \square \\ \square \square \end{array} = \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \square \square \end{array} \sim \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \square \square \end{array}$$

The diagram shows the amalgamation of composition D (two boxes, top blue, bottom red) and composition E (two boxes, top grey, bottom white) using the operation \circ_W with $W = \{\square\}$. The result is a composition of 15 boxes, which is shown to be equivalent to another composition of 15 boxes where the boxes are colored differently.

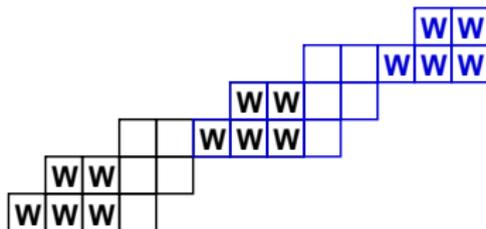
15 boxes: first of the non-RSvW examples
 If $W = \emptyset$, we get the regular compositions:

$$\begin{array}{c} \square \\ \square \square \end{array} \circ_{\emptyset} \begin{array}{c} \square \square \square \square \\ \square \square \square \end{array} = \begin{array}{c} \square \square \square \square \\ \square \square \square \square \\ \square \square \square \square \end{array}$$

The diagram shows the amalgamation of composition D (two boxes, top blue, bottom red) and composition E (two rows of four boxes, top white, bottom black) using the operation \circ_{\emptyset} . The result is a composition of 15 boxes, which is shown to be equivalent to another composition of 15 boxes where the boxes are colored differently.

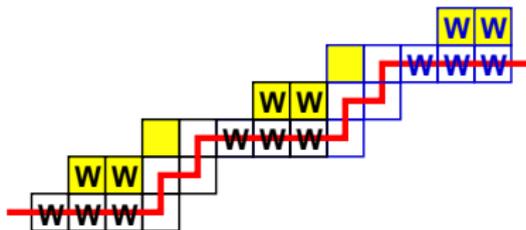
What are the results?

Construction of \overline{W} and \overline{O} :



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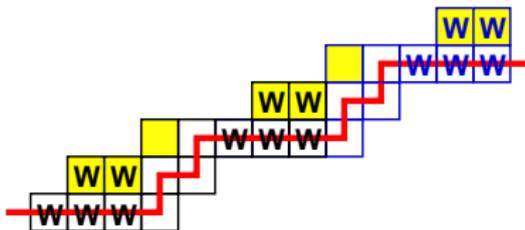
Construction of \overline{W} and \overline{O} :



Hypothesis 4. \overline{W} is never adjacent to \overline{O} .

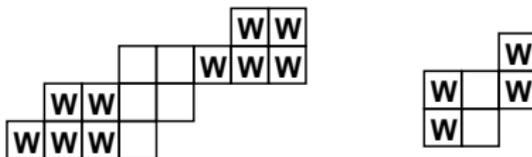
What are the results?

Construction of \overline{W} and \overline{O} :



Hypothesis 4. \overline{W} is never adjacent to \overline{O} .

Hypothesis 5. In $E = WOW$, at least one copy of W has just one cell adjacent to O .



What are the results?

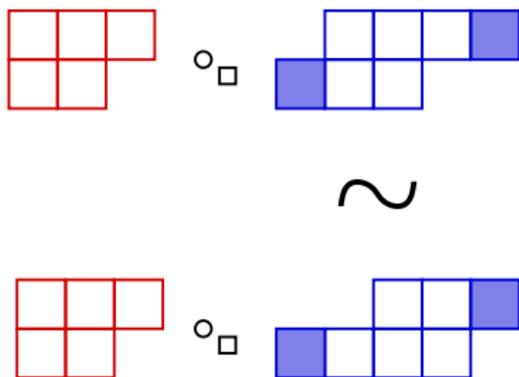
Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5. Then

$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$

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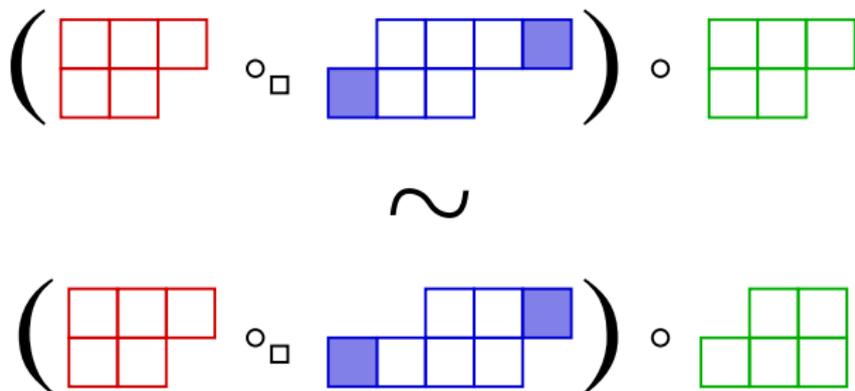
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$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$



is a skew equivalence with 145 boxes.

The key: An expression for $s_{D \circ_W E}$ in terms of s_D , s_E , $s_{\overline{W}}$, $s_{\overline{O}}$.

Proof of expression uses:

- ▶ Hamel-Goulden determinants.

Angèle M. Hamel and Ian P. Goulden:

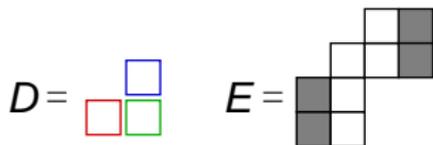
Planar decompositions of tableaux and Schur function determinants.

William Y. C. Chen, Guo-Guang Yan and Arthur L. B. Yang:

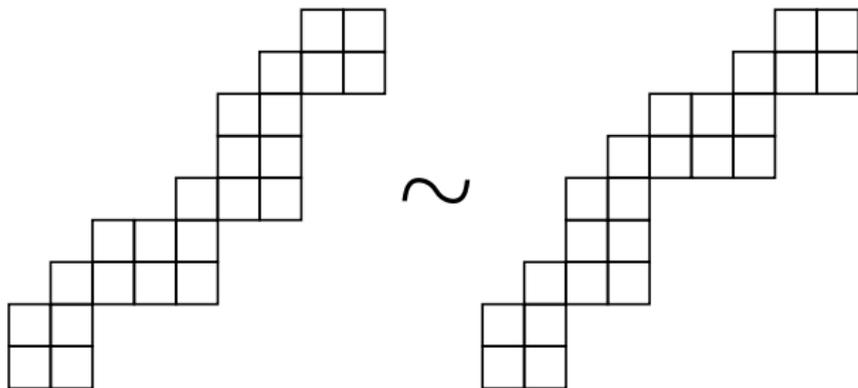
Transformations of border strips and Schur function determinants.

- ▶ Sylvester's Determinantal Identity.

- ▶ Removing Hypothesis 5 (at least one copy of W has just one cell adjacent to O).



$D \circ_W E$ has 23 boxes, and $D \circ_W E \sim D^* \circ_W E$:



(Software of Anders Buch, John Stembridge)

Main open problem

Theorem. [McN, van Willigenburg]

Skew diagrams E_1, E_2, \dots, E_r

$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5

E'_i and W'_i denote either E_i and W_i , or E_i^* and W_i^* .

Then

$$((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r.$$

Main open problem

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Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]

Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r ,

$$\begin{aligned} E &= ((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r, \text{ where} \end{aligned}$$

- $E_i = W_i O_i W_i$ satisfies Hypotheses 1-4 for all i ,
- E'_i and W'_i denote either E_i and W_i , or E_i^* and W_i^* .

Main open problem

Theorem. [McN, van Willigenburg]

Skew diagrams E_1, E_2, \dots, E_r

$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5

E'_i and W'_i denote either E_i and W_i , or E_i^* and W_i^* .

Then

$$((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r.$$

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- $E_i = W_i O_i W_i$ satisfies Hypotheses 1-4 for all i ,
- E'_i and W'_i denote either E_i and W_i , or E_i^* and W_i^* .

True when #boxes ≤ 20 .

- ▶ A definition of skew diagram composition. Encompasses the operation of BTvW and the three operations of RSvW.
- ▶ Theorem that generalizes all previous results.
In particular, explains the 6 missing equivalences from RSvW.
- ▶ Conjecture for necessary and sufficient conditions for $E \sim E'$.