

Positivity Questions for Cylindric Skew Schur Functions

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Slides and full paper available from
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- ▶ What are cylindric skew Schur functions?
- ▶ When are they Schur-positive?
- ▶ An expansion in terms of skew Schur functions
- ▶ A Schur-positivity conjecture

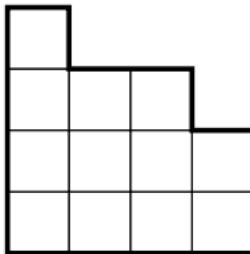
Schur functions

- ▶ Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

- ▶ Young diagram.

Example:

$$\lambda = (4, 4, 3, 1)$$



Schur functions

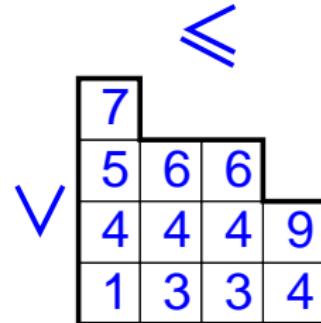
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- ▶ Young diagram.

Example:

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- ▶ Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \dots)$ is then defined by

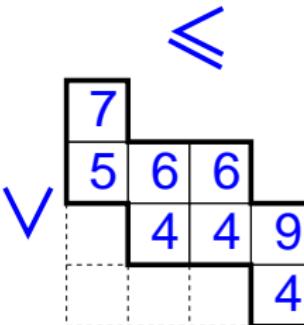
$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \dots .$$

Example

$$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \dots .$$

Skew Schur functions

- ▶ Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$
- ▶ μ fits inside λ .
- ▶ Young diagram.
Example:
 $\lambda/\mu = (4, 4, 3, 1)/(3, 1)$
- ▶ Semistandard Young tableau (SSYT)



The **skew** Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \dots)$ is then defined by

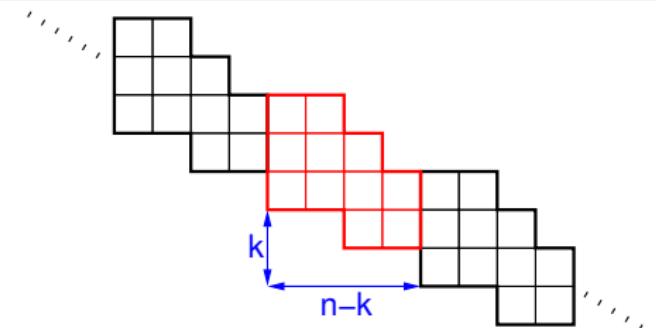
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Example

$$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \dots .$$

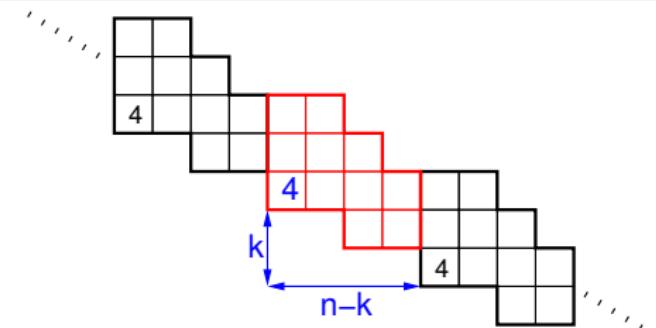
Cylindric skew Schur functions

- ▶ Infinite skew shape C
- ▶ Invariant under translation
- ▶ Identify (a, b) and $(a + n - k, b - k)$.



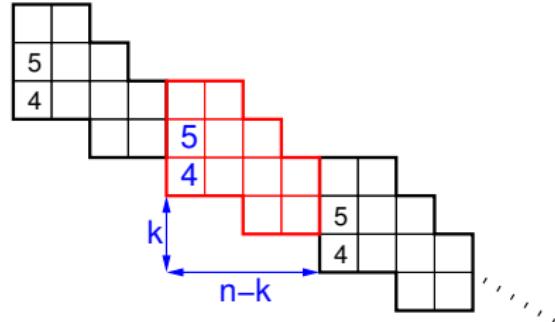
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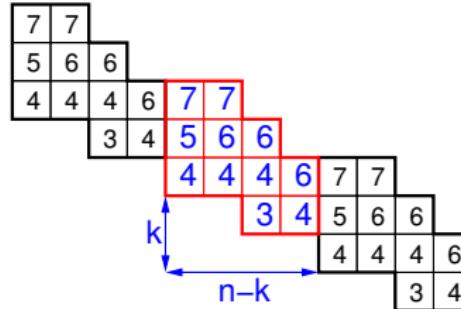
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Cylindric skew Schur functions

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- Entries weakly increase in each row
Strictly increase up each column
- Alternatively: SSYT with relations between entries in first and last columns
- Cylindric skew Schur function:**

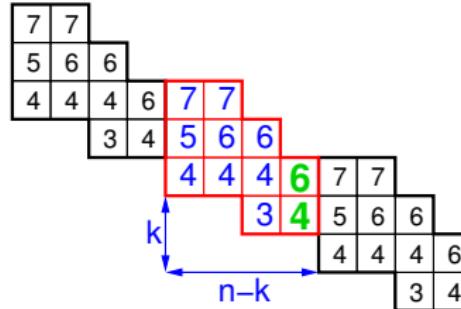
$$s_C(x) = \sum_T x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \dots .$$

$$\text{e.g. } s_C(x) = x_3 x_4^4 x_5 x_6^3 x_7^2 + \dots .$$

- s_C is a symmetric function

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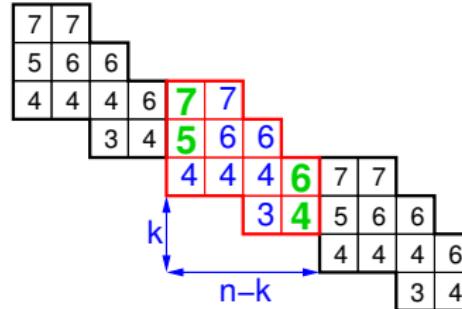
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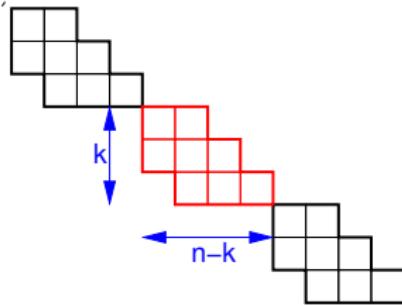
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Skew shapes are cylindric skew shapes...

... and so skew Schur functions are cylindric skew Schur functions.

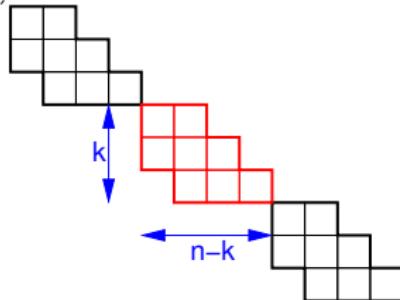
Example



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Example



- ▶ Gessel, Krattenthaler: “*Cylindric partitions*,” 1997.
- ▶ Bertram, Ciocan-Fontanine, Fulton: “*Quantum multiplication of Schur polynomials*,” 1999.
- ▶ Postnikov: “*Affine approach to quantum Schubert calculus*,” [math.CO/0205165](https://arxiv.org/abs/math.CO/0205165).
- ▶ Stanley: “*Recent developments in algebraic combinatorics*,” [math.CO/0211114](https://arxiv.org/abs/math.CO/0211114).

Motivation: Positivity of Gromov-Witten invariants

In $H^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c_{\mu\nu}^\lambda \sigma_\lambda.$$

In $QH^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\lambda \subseteq k \times (n-k)} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda.$$

$C_{\mu\nu}^{\lambda,d}$ = 3-point Gromov-Witten invariants

= # {rational curves of degree d in Gr_{kn} that meet $\tilde{\Omega}_\mu$, $\tilde{\Omega}_\nu$ and $\tilde{\Omega}_{\lambda^\vee}$ }.

Example

$$C_{\mu,\nu}^{\lambda,0} = c_{\mu\nu}^\lambda.$$

Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.

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Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.

“Fundamental open problem”: Find an algebraic or combinatorial proof of this fact.

Theorem (Postnikov)

$$s_{\mu/d/\nu}(x_1, \dots, x_k) = \sum_{\lambda \subseteq k \times (n-k)} C_{\mu\nu}^{\lambda,d} s_\lambda(x_1, \dots, x_k).$$

Conclusion: Want to understand the expansions of cylindric skew Schur functions into Schur functions.

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Corollary

$s_{\mu/d/\nu}(x_1, \dots, x_k)$ is Schur-positive.

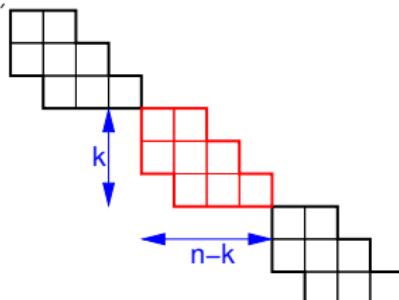
Known: $s_{\mu/d/\nu}(x_1, x_2, \dots) \equiv s_{\mu/d/\nu}(x)$ need not be Schur-positive.

Example

If $s_{\mu/d/\nu} = s_{22} + s_{211} - s_{1111}$, then $s_{\mu/d/\nu}(x_1, x_2, x_3)$ is Schur-positive.

(In general: $s_{\lambda}(x_1, \dots, x_k) \neq 0 \Leftrightarrow \lambda$ has at most k parts.)

When is a cylindric skew Schur function Schur-positive?



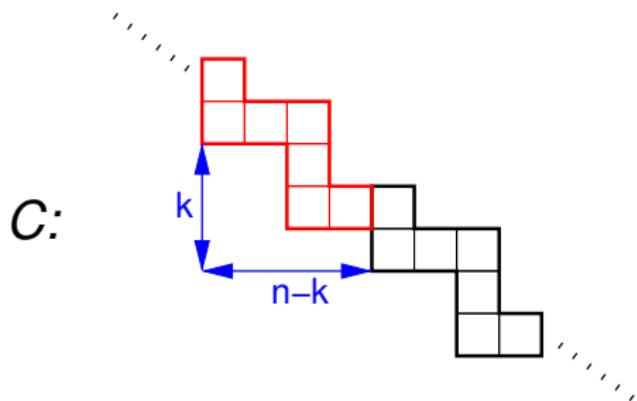
Theorem (McN.)

For any cylindric skew shape C ,

$$s_C(x_1, x_2, \dots) \text{ is Schur-positive} \Leftrightarrow C \text{ is a skew shape.}$$

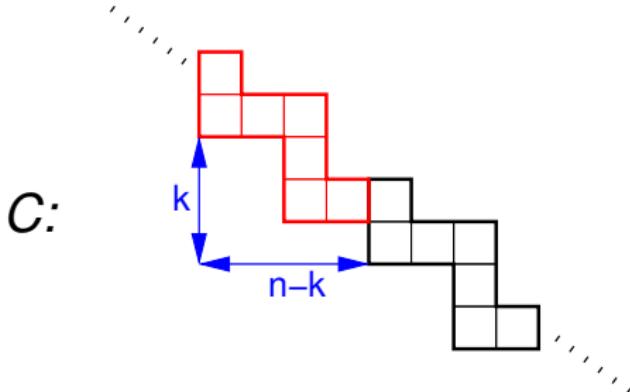
Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \dots)$ is **not** Schur-positive.

Example: cylindric ribbons



$$s_C(x_1, x_2, \dots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k, 1^k)} - s_{(n-k-1, 1^{k+1})} \\ + s_{(n-k-2, 1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

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Schur-positive in $k + 1$ variables.

Not Schur-positive in $\geq k + 2$ variables.

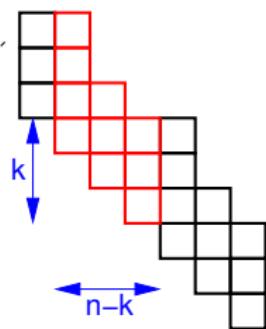
General cylindric skew shape: $\geq k + 2 + \ell$ variables.

Shapes in Postnikov's theorem: $\geq 2k + 1$ variables.

Formula: cylindric skew Schur functions as signed sums of skew Schur functions

Idea for formulation: Bertram, Ciocan-Fontanine, Fulton
Uses result of Gessel, Krattenthaler

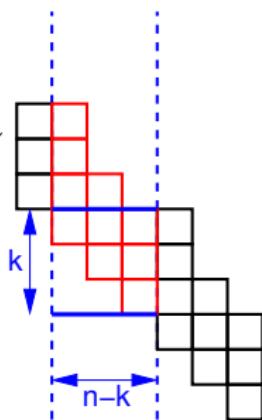
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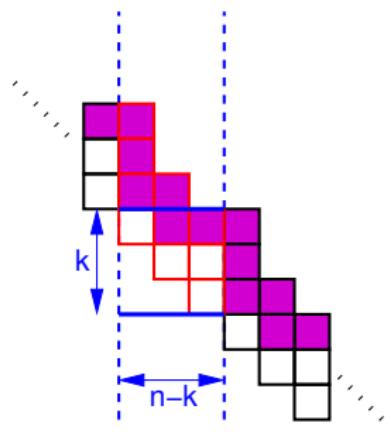
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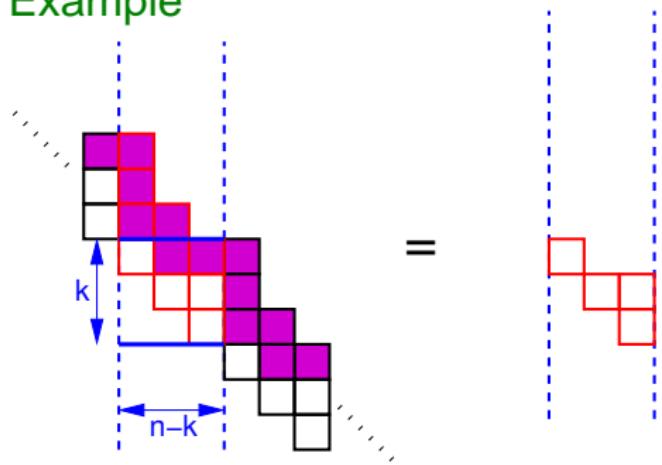
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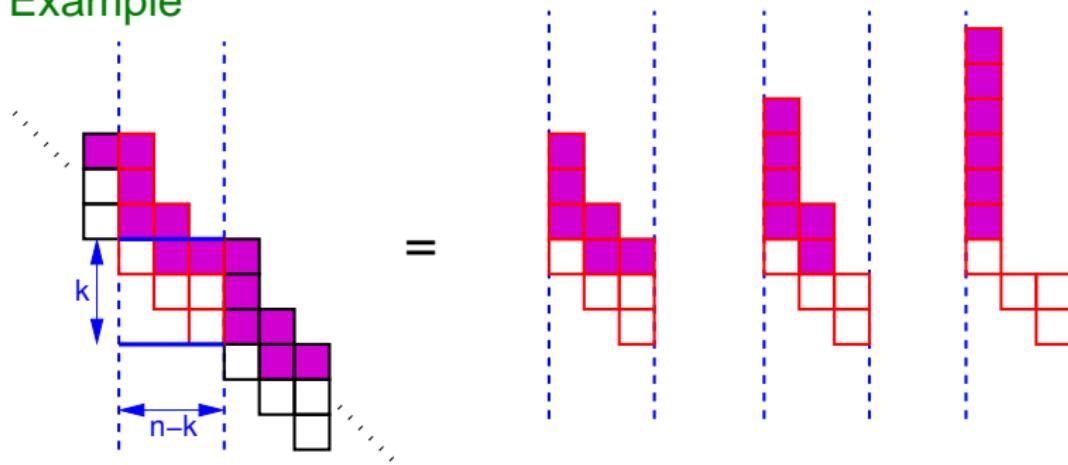
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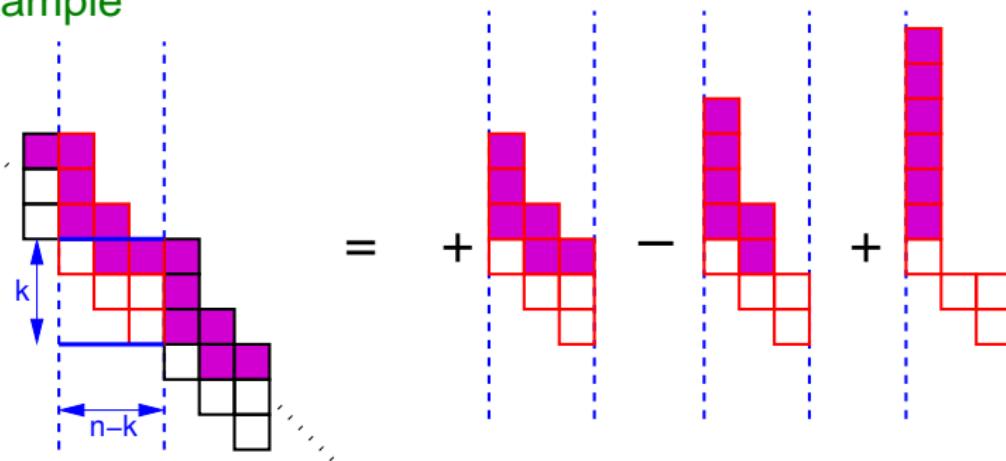
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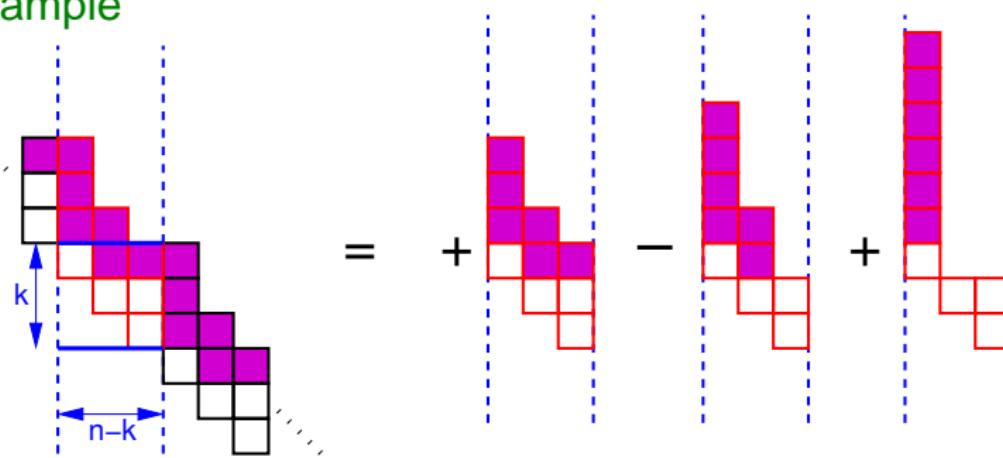
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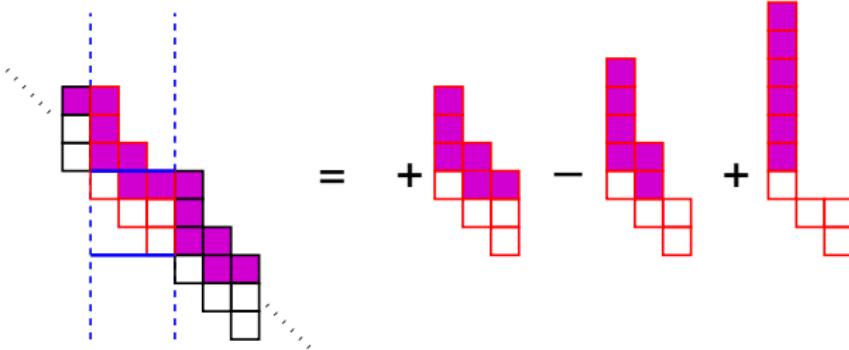
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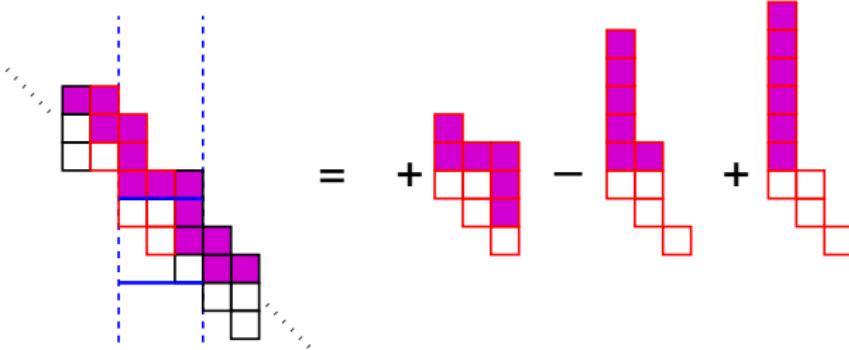


$$\begin{aligned} s_C &= s_{333211/21} - s_{3322111/21} + s_{33111111/21} \\ &= \textcolor{blue}{s_{3331}} + \textcolor{blue}{s_{3322}} + \textcolor{blue}{s_{33211}} + \textcolor{blue}{s_{322111}} + \textcolor{blue}{s_{31111111}} \\ &\quad - s_{222211} - s_{2221111} + s_{22111111} + s_{21111111}. \end{aligned}$$

Quick consequence: lots of skew Schur function identities

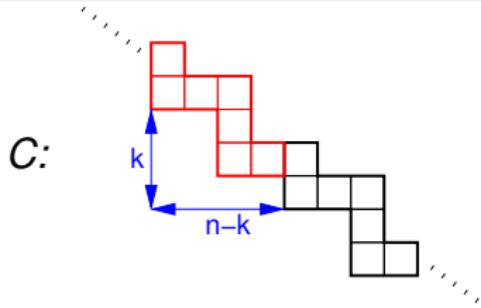
$$\begin{array}{c} \text{Diagram of a skew shape with blue dashed lines and red grid lines, partially filled with purple squares.} \\ = + - + \end{array}$$


A skew shape diagram consisting of several rows of boxes. Blue dashed vertical lines are positioned at the start of the first, third, and fourth rows. Red grid lines run from the top-left to the bottom-right through the boxes. Some boxes are filled with purple, while others are white. To the right of the diagram is an equals sign followed by three terms separated by plus and minus signs.

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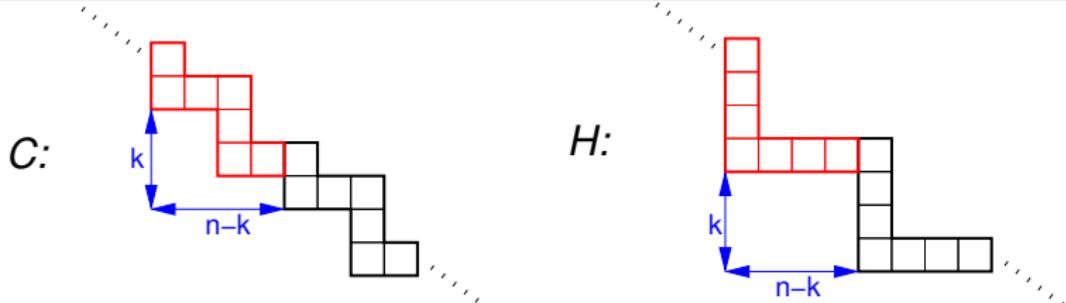
A second skew shape diagram, similar in structure to the first, with blue dashed lines and red grid lines. It contains purple-filled boxes and white boxes. To its right is an equals sign followed by three terms separated by plus and minus signs.

Shouldn't cylindric skew Schur functions be Schur-positive *in some sense?*



$$s_C(x_1, x_2, \dots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k, 1^k)} - s_{(n-k-1, 1^{k+1})} \\ + s_{(n-k-2, 1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

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In fact,

$$s_C(x_1, x_2, \dots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_H.$$

s_C : cylindric skew Schur function

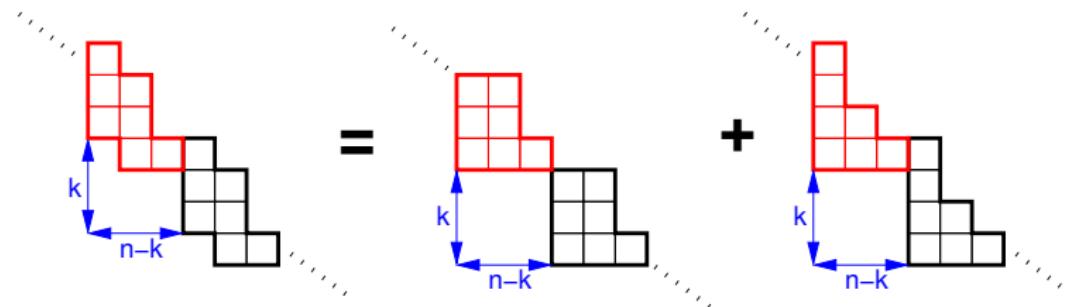
s_H : cylindric Schur function

We say that s_C is **cylindric Schur-positive**.

A Conjecture

$$\begin{array}{c} \text{Diagram 1: A skew shape with a red top-left corner and black bottom-right corner. A blue arrow labeled 'k' points down from the top-left to the boundary, and a blue arrow labeled 'n-k' points right from the bottom-left to the boundary.} \\ = \\ \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but the red region is shifted up by one row. A blue arrow labeled 'k' points down from the top-left to the boundary, and a blue arrow labeled 'n-k' points right from the bottom-left to the boundary.} \\ + \\ \begin{array}{c} \text{Diagram 3: Similar to Diagram 1, but the red region is shifted up by one row. A blue arrow labeled 'k' points down from the top-left to the boundary, and a blue arrow labeled 'n-k' points right from the bottom-left to the boundary.} \end{array} \end{array} \end{array}$$

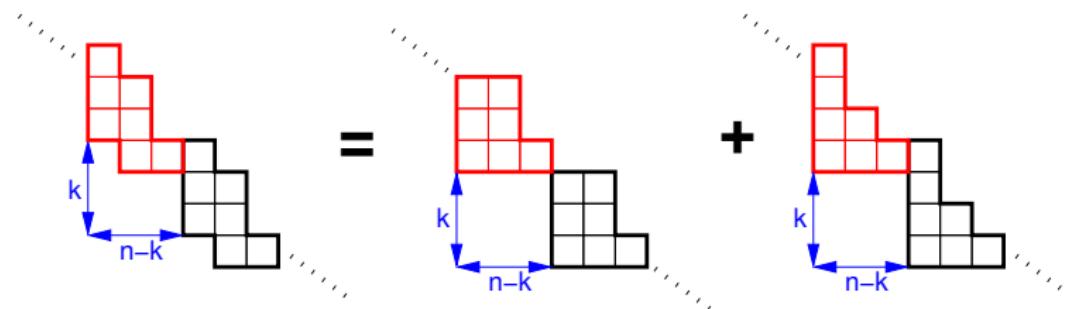
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Theorem (McN.)

The cylindric Schur functions corresponding to a given translation $(-n+k, +k)$ are linearly independent.

Theorem (McN.)

If s_C can be written as a linear combination of cylindric Schur functions with the same translation as C , then s_C is cylindric Schur-positive.

- ▶ Classification of those cylindric skew Schur functions that are Schur-positive.
- ▶ Full knowledge of negative terms in Schur expansion of ribbons.
- ▶ Expansion of any cylindric skew Schur function into skew Schur functions.
- ▶ Conjecture and evidence that every cylindric skew Schur function is cylindric Schur-positive.