The Schur-Positivity Poset

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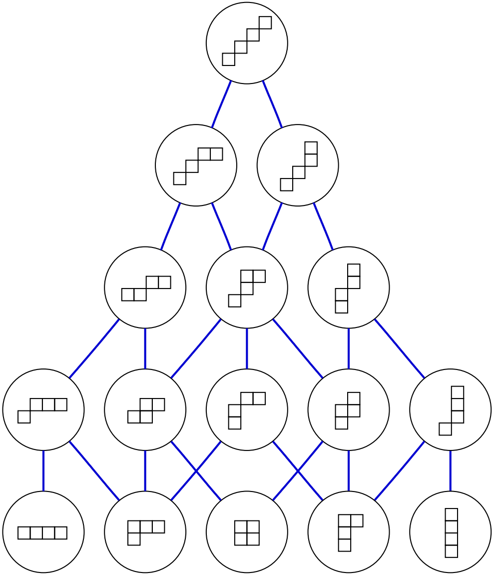
Slides and papers available from
www.facstaff.bucknell.edu/pm040/

Outline

- Symmetric functions background
- Definition of the Schur-positivity poset
- Some known and unknown properties
- Focus on necessary conditions for $A \leq_s B$.

Preview

$n = 4$



What are symmetric functions?

Definition. A symmetric polynomial is a polynomial that is invariant under any permutation of its variables $x_1, x_2, \ldots x_n$.

Example.

- $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$ is a symmetric polynomial in x_1, x_2, x_3.

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  is a symmetric polynomial in x_1, x_2, x_3.

Definition. A symmetric function is a formal power series that is invariant under any permutation of its (infinite set of) variables $x = (x_1, x_2, \ldots)$.

Examples.

- $(x_1 + x_2 + x_3 + \cdots)(x_1^2 + x_2^2 + x_3^2 + \cdots)$ is a symmetric function.
- $\sum_{i<j} x_i^2 x_j$ is not symmetric.

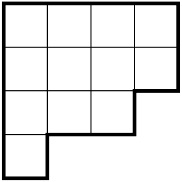
Fact: The symmetric functions (over $\mathbb{Q}$, say) form an algebra.

Schur functions

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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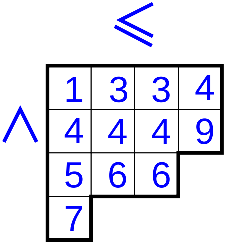
- Semistandard Young tableau (SSYT)



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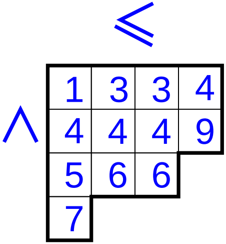


The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

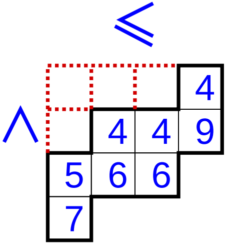
Example.

$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

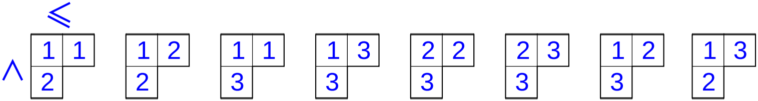
$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example.

$s_{4431/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$

Examples.

$$s_{21}(x_1, x_2, x_3) =$$



$$x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3$$

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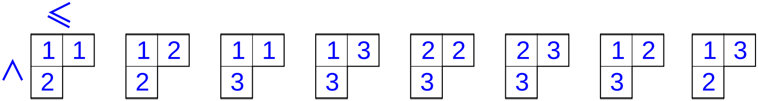

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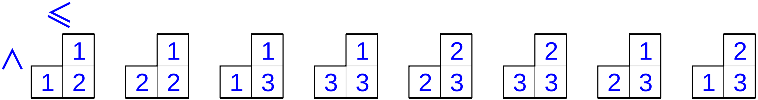

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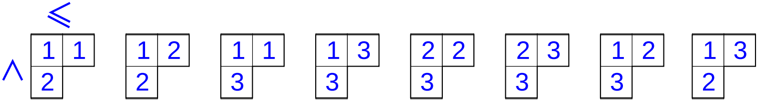

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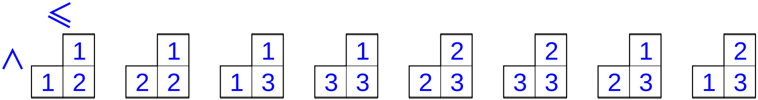

$$s_{21}(x) = s_{22/1}(x) = \sum_{i \neq j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k$$

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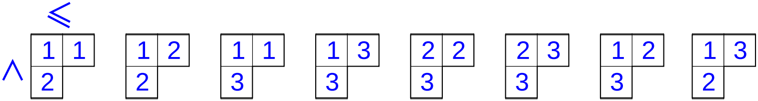

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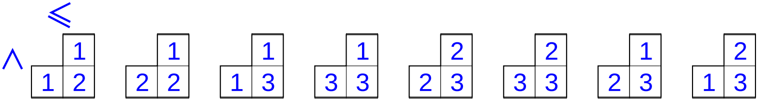

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Fact: Skew Schur functions are symmetric functions.

Question: Why do we care about Schur functions?

s_λ and $c^\lambda_{\mu\nu}$ are superstars!

Fact: The Schur functions form a basis for the algebra of symmetric functions.

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

$c^\lambda_{\mu\nu}$: Littlewood-Richardson coefficients

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$c^\lambda_{\mu\nu}$: Littlewood-Richardson coefficients

1. Multiplicative coefficients: $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$.
2. Representation Theory of S_n: $\chi^\mu \cdot \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda$.
3. Representations of $\mathrm{GL}(n, \mathbb{C})$:
   $s_\lambda(x_1, \ldots, x_n) =$ the character of the irreducible rep. V^λ.
4. Algebraic Geometry: Schubert classes σ_λ form a linear basis for $H^*(\mathrm{Gr}_{kn})$.

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c^\lambda_{\mu\nu} \sigma_\lambda.$$

There's more!

4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

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There's more!

4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively? When $c^{\lambda}_{\mu\nu} > 0$. (Heckman, Klyachko, Knutson, Tao.)

By 2 we get:

$$c^{\lambda}_{\mu\nu} \geq 0.$$ (Your take-home fact!)

Consequences:

- We say that $s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}$ is a Schur-positive function, i.e., coefficients in Schur expansion are all non-negative.
- Want a combinatorial proof: "They must count something simpler!"

Littlewood-Richardson Rule

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

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Littlewood-Richardson rule [Littlewood-Richardson 1934, Schützenberger 1977, Thomas 1974].

$c_{\mu\nu}^{\lambda}$ is the number of SSYT of shape λ/μ and content ν whose reverse reading word is a ballot sequence.

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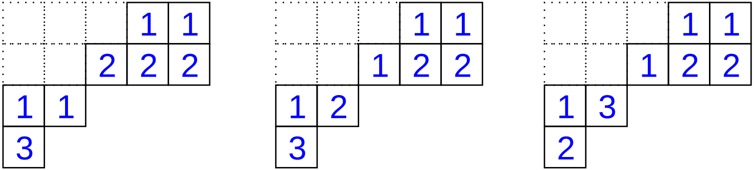
Example.

When $\lambda = (5, 5, 2, 1), \mu = (3, 2), \nu = (4, 3, 1)$, we get $c^{\lambda}_{\mu\nu} = 2$.



11222113 No   11221213 Yes   11221312 Yes

The story so far

Consequences of the Littlewood Richardson rule :

- A combinatorial proof that $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$ is Schur-positive.
- A way to calculate $c^\lambda_{\mu\nu}$.

(Natural connections between Schur-positivity and representation theory.)

Summary so far:

- Schur functions form important basis for symmetric functions.
- Skew Schur functions indexed by skew shapes.
- Skew Schur functions are Schur-positive.
- Littlewood-Richardson rule gives a way to determine the Schur expansion of a skew Schur function.

Schur-positivity order

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

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When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

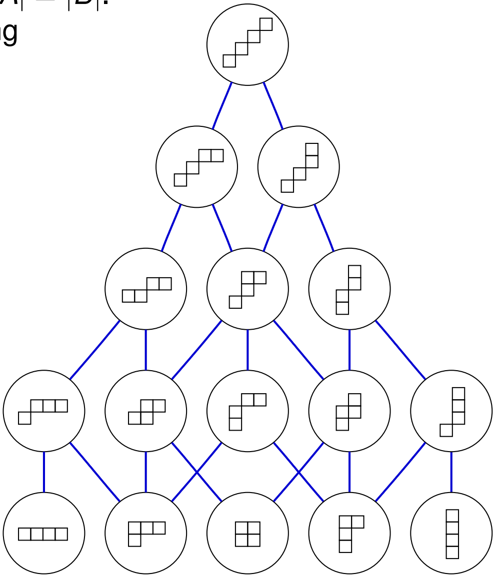
Definition. Let A, B be skew shapes. We say that

$$A \leq_s B \quad \text{if} \quad s_B - s_A \quad \text{is Schur-positive.}$$

Goal: Characterize the Schur-positivity order $\leq_s$ in terms of skew shapes.

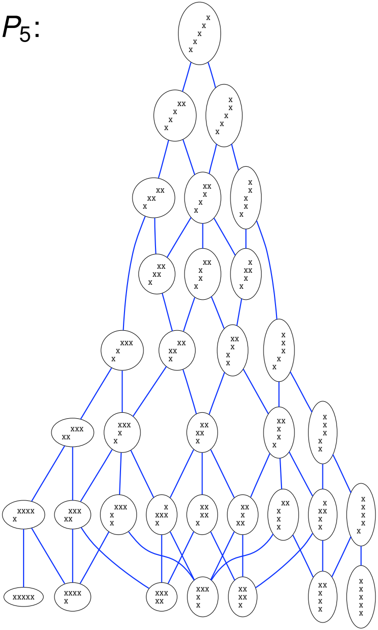
Example of a Schur-positivity poset

If $A \leq_s B$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

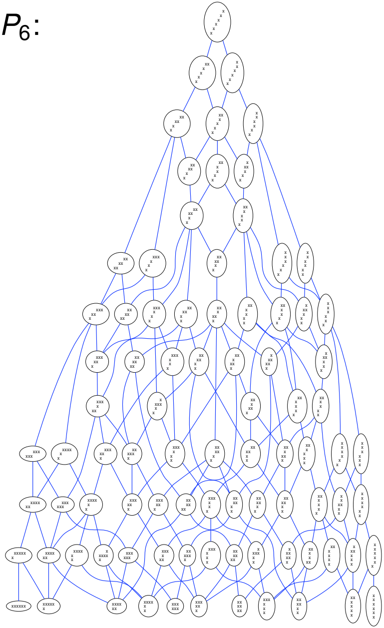


More examples

P_5:

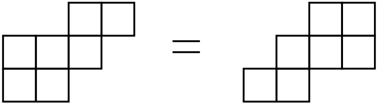
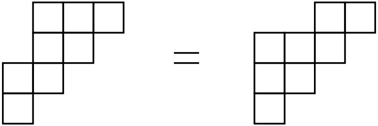
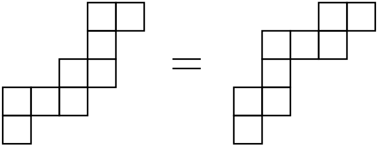


P_6:



Known properties: first things first

$\leq_s$ is not yet anti-symmetric. So identify skew shapes such as

1. 
2. 
3. 

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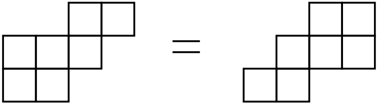

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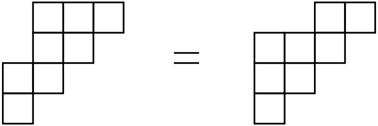

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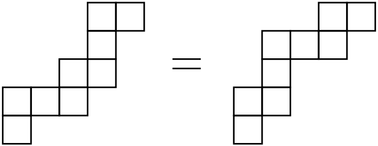

Definition.

A ribbon is a connected skew shape containing no 2×2 rectangle.

Known properties: skew Schur equality

Question: When is $s_A = s_B$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

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Complete classification of equality of ribbon Schur functions

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  Complete classification of equality of ribbon Schur functions
- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)

Enough for our purposes: we can consider P_n to be a poset.

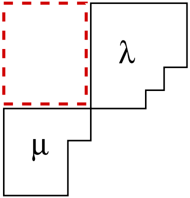
Open Problem: Find necessary and sufficient conditions on A and B for $s_A = s_B$.

Known properties: Sufficient conditions

Sufficient conditions for $A \leq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- ...

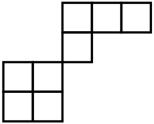
Note: $s_\lambda s_\mu$ is a special case of s_A.



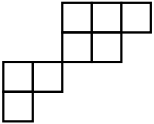
Theorem [LPP]. For skew shapes λ and μ,

$$s_\lambda s_\mu \leq_s s_{\lambda\cup\mu} s_{\lambda\cap\mu}$$

Example.



$$\leq_s$$



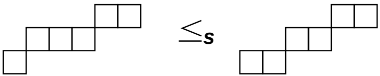
Known properties: special classes of skew shapes

Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

$$\lambda_1 + \cdots \lambda_i \leq \mu_1 + \cdots \mu_i \text{ for all } i.$$

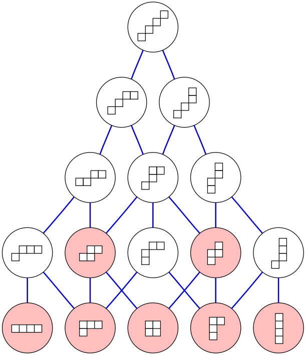
- Macdonald's "Symmetric functions and Hall polynomials": For horizontal strips, $A \leq_s B$ if and only if

  row lengths of $A \succcurlyeq$ row lengths of B

  P_n restricted to horizontal strips: (dual of the) dominance lattice.

Unknown property: maximal connected skew shapes



Question: What are the maximal elements of P_n among the connected skew shapes?

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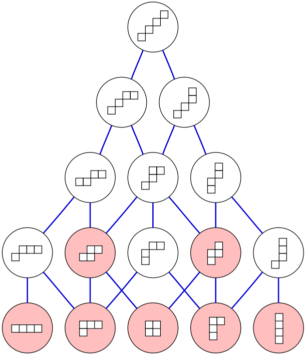
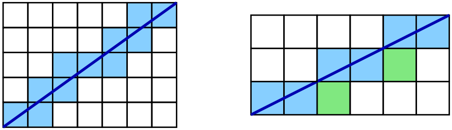


Question: What are the maximal elements of P_n among the connected skew shapes?

Conjecture [McN., Pylyavskyy]. For each $r = 1, \ldots, n$, there is a unique maximal connected element with r rows, namely the ribbon marked out by the diagonal of an r-by-$(n-r+1)$ box.

Examples.



Necessary conditions

Question: Suppose $A \leq_s B$ (i.e. $s_B - s_A$ is Schur-positive). Then what can we say about the shapes A and B?

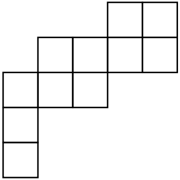
Such necessary conditions for $A \leq_s B$ give us a way to show that $C \not\leq_s D$.

Example. If $A \leq_s B$, then $|A| = |B|$.

Classical necessary conditions

Notation. For a skew shape A, let rows(A) denote the partition of row lengths of A. Define cols(A) similarly.

Example. rows$(A) = 43211$, cols$(A) = 32222$.



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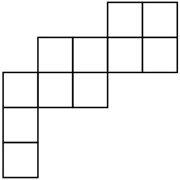
$$\begin{aligned} s_A &= s_{551} + s_{542} + 2s_{5411} + s_{533} + 2s_{5321} + s_{53111} \\ &\quad + s_{52211} + s_{4421} + s_{44111} + s_{4331} + s_{43211}. \\ \text{support}(A) &= \{551, 542, 5411, 533, 5321, 53111, \\ &\quad 52211, 4421, 44111, 4331, 43211\}. \end{aligned}$$



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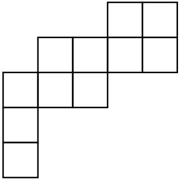
Proposition. In the Schur expansion of A:

- $\text{rows}(A)$ is the **least** dominant partition in the support of A.
- $(\text{cols}(A))^t$ is the **most** dominant partition in the support of A.



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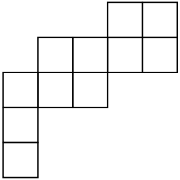
"Proof":



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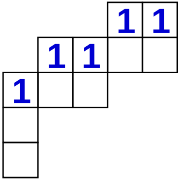

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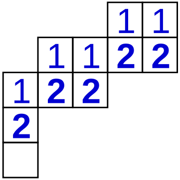

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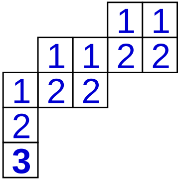

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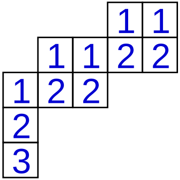

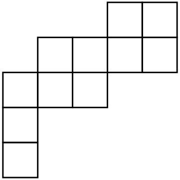

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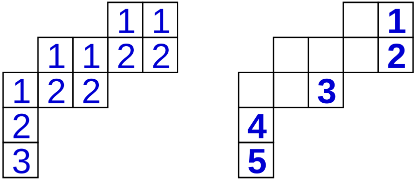

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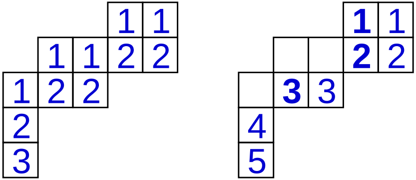

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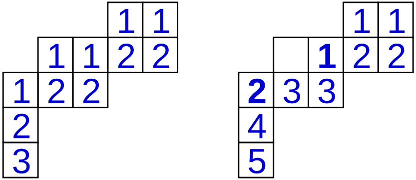

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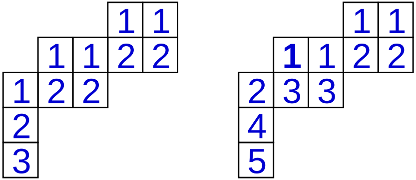

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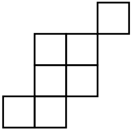
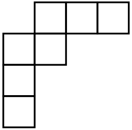
$C =$  $D =$ 

$\text{rows}(C) = 2221 \prec 3211 = \text{rows}(D)$.

Thus $C \not\leq_s D$.

Key definitions: generalize rows(A) and cols(A)

Definitions [Reiner, Shaw, van Willigenburg]. For a skew shape A, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.

Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

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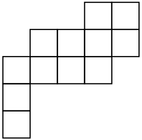
$A =$ 

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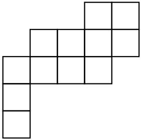
$A =$ 

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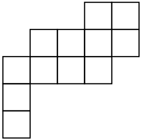
$A =$ 

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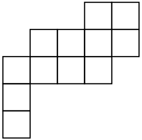
$A =$ 

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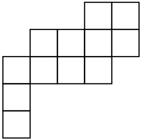
$A =$ 

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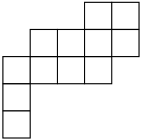
$A =$ 

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Necessary conditions

Theorem [RSvW]. Let A and B be skew shapes. If $s_A = s_B$, then

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Corollary. Let A and B be skew shapes. If $\text{support}(A) = \text{support}(B)$, then

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Relating $\text{rows}_k(A)$ and $\text{cols}_k(A)$

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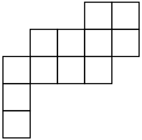
$A =$ 

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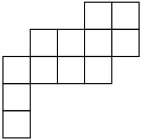
$A =$ 

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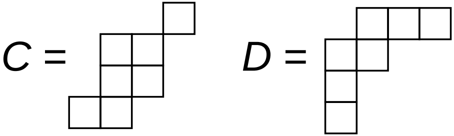

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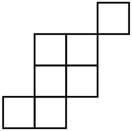
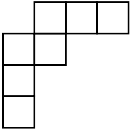
$C =$  $D =$ 

$\text{rows}(C) = 2221 \prec 3211 = \text{rows}(D)$. Thus $C \not\leq_s D$.

$\text{rows}_2(C) = 21 \succ 111 = \text{rows}_2(D)$. Thus $D \not\leq_s C$.

Outlook

- Instead of looking at the Schur-positivity poset, could look at the support containment poset; it seems to have more structure.
- Almost nothing is known about the covering relations in P_n.
- Why restrict to skew Schur functions? Could try:
  - Stanley symmetric functions
  - Hall-Littlewood polynomials
  - LLT-polynomials
  - Cylindric Schur functions
  - Skew Grothendieck polynomials
  - Poset quasisymmetric functions
  - Wave Schur functions
  - ...