Comparing skew Schur functions: a quasisymmetric perspective

Peter McNamara

Bucknell University

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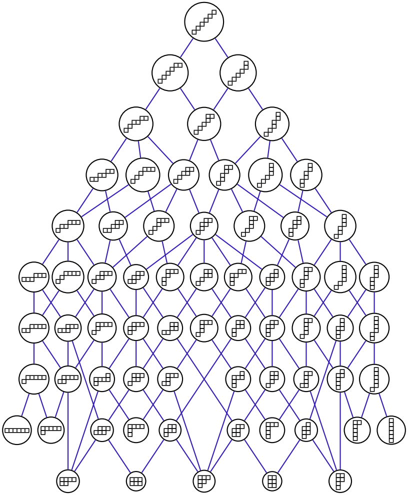
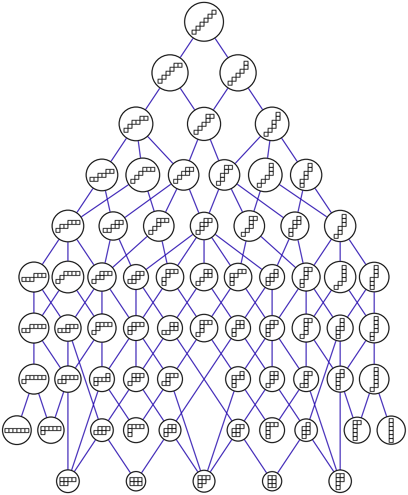
Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

Outline

- The background story: the equality question
- Conditions for Schur-positivity
- Quasisymmetric insights and the big conjecture
- Relationship to other (quasi)symmetric bases



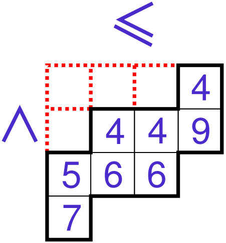
F-support containment



Row overlap reverse dominance

Skew Schur functions

- Skew shape A
- e.g. $A = 4431/31$
- Semistandard Young tableau (SSYT)



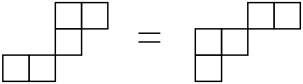
The skew Schur function s_A in the variables $x = (x_1, x_2, \ldots)$ is then defined by

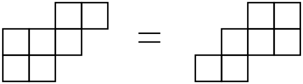
$$s_A = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

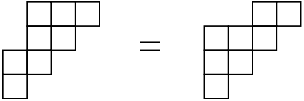
Example.

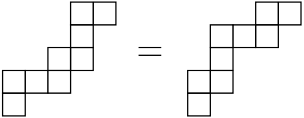
$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$

Question. When is $s_A = s_B$?

1. 

2. 

3. 

4. 

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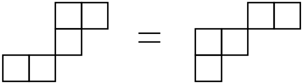
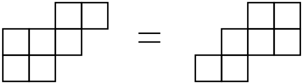
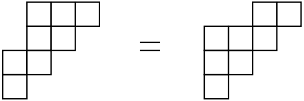
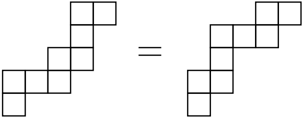
1. 

2. 

3. 

4. 

Definition.
A ribbon is a connected skew shape containing no 2×2 rectangle.

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  Complete classification of equality of ribbon Schur functions.

- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)
- Christian Gutschwager (2008) solved multiplicity-free case

Open Problem. Find necessary and sufficient conditions on A and B for $s_A = s_B$.

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Definition [Reiner, Shaw, van Willigenburg]. For a skew shape A, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.
Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

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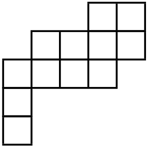
$A =$ 

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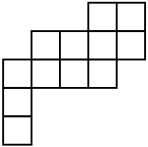
$A =$ 

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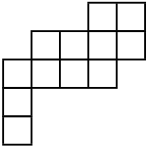
$A =$ 

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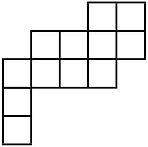
$A =$ 

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- $\text{rows}_3(A) = 11$.

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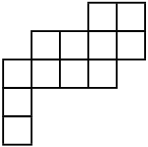
$A =$ 

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- $\mathrm{rows}_3(A) = 11$.
- $\mathrm{rows}_k(A) = \emptyset$ for $k > 3$.

Necessary conditions for equality

Theorem [RSvW]. Let A and B be skew shapes. If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \ \text{ for all } k.$$

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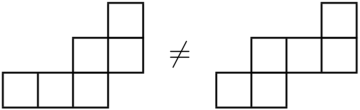

Schur-positivity order

Our interest: inequalities.

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

Schur-positivity order

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When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

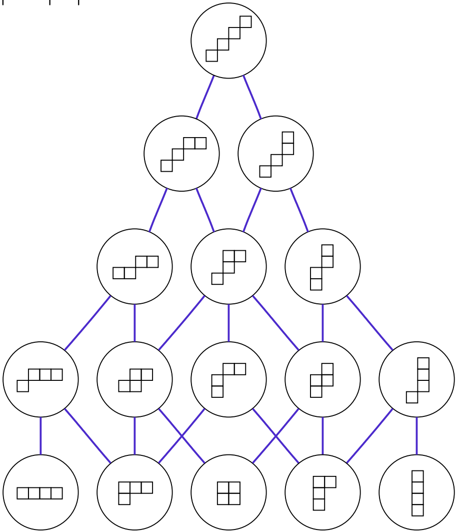
Definition. Let A, B be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

Original goal: characterize the Schur-positivity order $\geq_s$ in terms of skew shapes.

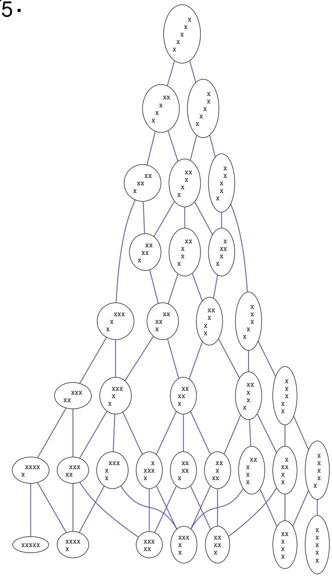
Example of a Schur-positivity poset

If $B \leq_s A$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

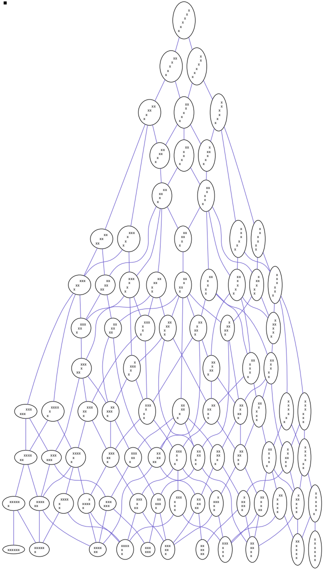


More examples

P_5:



P_6:



Known properties: Sufficient conditions

Sufficient conditions for $A \geq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- McN., Steph van Willigenburg (2009, 2012)
- ...

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Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

$$\lambda_1 + \cdots \lambda_i \leq \mu_1 + \cdots \mu_i \text{ for all } i.$$

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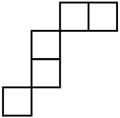
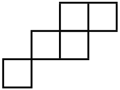
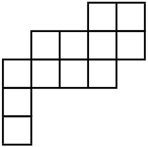
$A =$  $B =$ 

$$s_A = s_{41} + s_{32} + 2s_{311} + s_{221} + s_{2111}$$
$$s_B = s_{41} + 2s_{32} + s_{311} + s_{221}$$

So $s_A - s_B$ is not Schur-positive but $\text{supp}_s(A) \supseteq \text{supp}_s(B)$.

Equivalent to row overlap conditions

Let $\text{rects}_{k,\ell}(A)$ denote the number of $k \times \ell$ rectangular subdiagrams contained inside A.

$A =$  $\text{rects}_{3,1}(A) = 2$, $\text{rects}_{2,2}(A) = 3$, etc.

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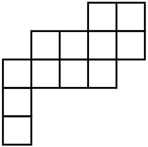
$A =$  rects$_{3,1}(A) = 2$, rects$_{2,2}(A) = 3$, etc.

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Summary so far

$$\boxed{s_A - s_B \text{ is Schur-pos.}} \quad \Rightarrow \quad \boxed{\text{supp}_s(A) \supseteq \text{supp}_s(B)} \quad \Rightarrow \quad \boxed{\begin{array}{l} \text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k \\ \text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell \\ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$$

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MIND THE GAP

Converse is very false.

$s_A - s_B$ is Schur-pos. $\Rightarrow$ $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$ $\Rightarrow$

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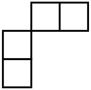
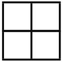
$A =$ 

$B =$ 

$s_A = s_{31} + s_{211}$ $\qquad$ $s_B = s_{22}$

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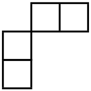
$A =$ 

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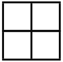
$B =$ 

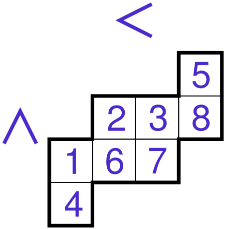
$s_B = s_{22}$

New Goal: Find weaker algebraic conditions on A and B that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?

F-basis of quasisymmetric functions

- Skew shape A.
- SYT T of A.



F-basis of quasisymmetric functions

- Skew shape A.
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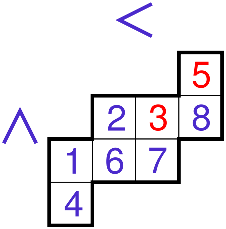
- Descent set $S(T) = \{3, 5\}$.
- Descent composition $\text{comp}(T) = 323$.



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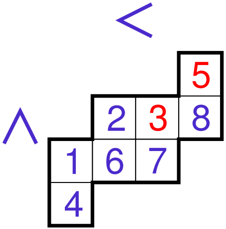

Then s_A expands in the basis of fundamental quasisymmetric functions as

$$s_A = \sum_{\text{SYT } T} F_{\mathrm{comp}(T)}.$$

Example.

$$s_{4431/31} = F_{323} + \cdots .$$

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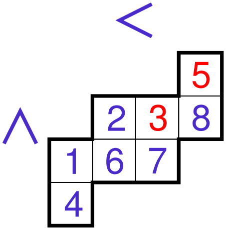

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Facts.

- The F form a basis for the quasisymmetric functions.
- So notions of F-positivity and F-support make sense.
- Schur-positivity implies F-positivity.
- $\text{supp}_s(A) \supseteq \text{supp}_s(B)$ implies $\text{supp}_F(A) \supseteq \text{supp}_F(B)$

New results: filling the gap

Theorem. [McN. (2013)]

$s_A - s_B$ is Schur-pos.	$\Rightarrow$	$\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$		
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$	$\Rightarrow$	$\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$ $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$ $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$

Theorem. [McN. (2013)]

$s_A - s_B$ is Schur-pos.	$\Rightarrow$	$\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$		
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$	$\Longleftrightarrow$	$\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$ $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$ $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$

Conjecture. The rightmost implication is iff.

New results: filling the gap

Theorem. [McN. (2013)]

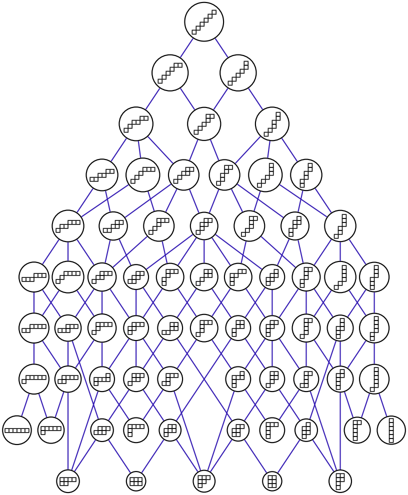
$s_A - s_B$ is Schur-pos.	$\Rightarrow$	$\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$				
$\Downarrow$		$\Downarrow$				
$s_A - s_B$ is F-positive	$\Rightarrow$	$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$	$\Longleftrightarrow$	$\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$ $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$ $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$		

Conjecture. The rightmost implication is iff.

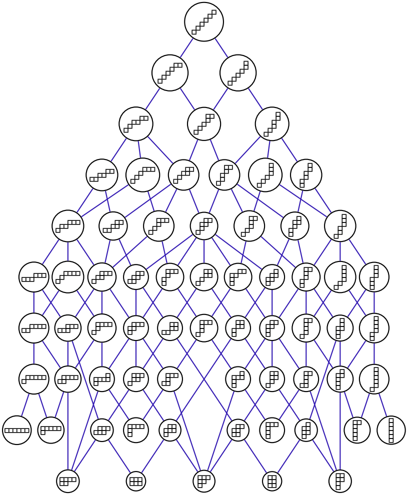
Evidence. Conjecture is true for:

- $n \leq 12$;
- horizontal strips;
- F-multiplicity-free skew shapes (as determined by Christine Bessenrodt and Steph van Willigenburg (2013));
- ribbons whose rows all have length at least 2.

$n = 6$ example



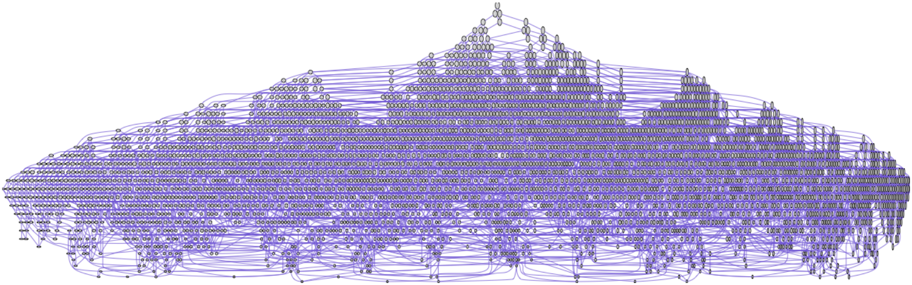
F-support containment



Row overlap reverse dominance

$n = 12$

$n = 12$ case has 12,042 edges.



Adding other bases

$s_A - s_B$ is D-positive

$\Downarrow$

$s_A - s_B$ is Schur-pos. $s_A - s_B$ is S-positive	$\Rightarrow$	$\text{supp}_D(A) \supseteq \text{supp}_D(B)$ $\text{supp}_s(A) \supseteq \text{supp}_s(B)$ $\text{supp}_S(A) \supseteq \text{supp}_S(B)$		
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\text{supp}_F(A) \supseteq \text{supp}_F(B)$	$\overset{?}{\Leftarrow}$ $\Rightarrow$	$\text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k$ $\text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell$ $\text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B)\ \forall k, \ell$
$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is M-positive	$\Rightarrow$	$\text{supp}_M(A) \supseteq \text{supp}_M(B)$		

Adding other bases

$s_A - s_B$ is D-positive

$\Downarrow$

$s_A - s_B$ is Schur-pos.
$s_A - s_B$ is S-positive $\Rightarrow$ $\mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B)$, $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$, $\mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B)$

$\Downarrow$ $\qquad$ $\Downarrow$

$s_A - s_B$ is F-positive $\Rightarrow$ $\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$ $\overset{?}{\Leftarrow}$ $\Rightarrow$ $\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$, $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$, $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$

$\Downarrow$ $\qquad$ $\Downarrow$

$s_A - s_B$ is M-positive $\Rightarrow$ $\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)$

Thanks! Merci!