About half the talk and almost all the mathematics was done on the blackboard and is not shown in these slides.

The Art of Double Counting

Peter McNamara
Bucknell University

Student Colloquium Series

10th October 2013

Slides available from
`www.facstaff.bucknell.edu/pm040/`

The Art of Using Different Counts for the Same Thing

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Proofs that Really Count:
The Art of Combinatorial Proof

Peter McNamara
Bucknell University

Student Colloquium Series

10th October 2013

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So what is combinatorics?

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count the number of elements in some collection of objects (i.e. enumerative questions).

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Vague: combinatorics is the study of finite sets.

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count the number of elements in some collection of objects (i.e. enumerative questions).

Combinatorics is an honest subject. No adèles, no sigma-algebras. You count balls in a box, and you either have the right number or you haven’t....Don’t get the wrong idea—combinatorics is not just putting balls into boxes. Counting finite sets can be a highbrow undertaking, with sophisticated techniques.

– Gian-Carlo Rota

What is a combinatorial proof?

Two types:

1. Bijective proofs: show that two sets have the same size.
2. Double counting proofs.

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1. Bijective proofs: show that two sets have the same size.
2. Double counting proofs.

Goal for the rest of the talk: convince you that by counting the same set in two different ways, we can give simple proofs of some beautiful identities.

Claim: combinatorial proofs tell you why something is true.

Pascal's triangle

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$$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

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$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

$$\binom{6}{0} = 1 \quad \binom{6}{1} = 6 \quad \binom{6}{2} = 15 \quad \binom{6}{3} = 20 \quad \binom{6}{4} = 15 \quad \binom{6}{5} = 6 \quad \binom{6}{6} = 1$$

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Application:

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6.$$

Algebraic proof of (4)

$$(1+x)^{2n} = ((1+x)^n)^2.$$

Now expand both sides using the Binomial Theorem.

$$\binom{2n}{0}x^0+\binom{2n}{1}x^1+\cdots+\binom{2n}{2n}x^{2n} = \left(\binom{n}{0}x^0+\binom{n}{1}x^1+\cdots+\binom{n}{n}x^n\right)^2$$

If these two sides are equal, the coefficients must match up. Extract the coefficient of x^n on both sides to get

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}.$$

Applying (1) gives

$$\binom{2n}{n} = \binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2,$$

as required.

Shameless plug

Math 319 in the spring: Combinatorics

Prereq: Math 280