Symmetric Functions and Cylindric Schur Functions

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Bucknell Mathematics Seminar
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Slides and papers available from
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What is algebraic combinatorics anyhow?

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The biggest open problem in algebraic combinatorics:

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Combinatorics that takes its problems, or its tools, from commutative algebra, algebraic geometry, algebraic topology, representation theory, etc.

Outline

- Symmetric functions
- Schur functions and Littlewood-Richardson coefficients
- Motivation for cylindric skew Schur functions
- Exposition of results

What are symmetric functions?

Definition

A symmetric polynomial is a polynomial that is invariant under any permutation of its variables $x_1, x_2, \dots x_n$.

Example

- $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$ is a symmetric polynomial in x_1, x_2, x_3.

Definition

A symmetric function is a formal power series that is invariant under any permutation of its (infinite set of) variables $x = (x_1, x_2, \dots)$.

Examples

- $\sum_{i \neq j} x_i^2 x_j$ is a symmetric function.
- $\sum_{i < j} x_i^2 x_j$ is not symmetric.

Bases for the symmetric functions

Fact: The symmetric functions form a vector space.
What is a possible basis?

- Monomial symmetric functions: Start with a monomial:

$$x_1^7 x_2^4$$

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$$x_1^7x_2^4 + x_1^4x_2^7 + x_1^7x_3^4 + x_1^4x_3^7 + \cdots .$$

Given a *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, e.g. $\lambda = (7, 4)$,

$$m_\lambda = \sum_{\substack{i_1,\ldots,i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \ldots x_{i_\ell}^{\lambda_\ell}.$$

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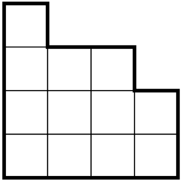
- Elementary symmetric functions, e_λ.
- Complete homogeneous symmetric functions, h_λ.
- Power sum symmetric functions, p_λ.

Typical questions: Prove they are bases, convert from one to another, ...

Cauchy, 1815.

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
- Young diagram.
  Example: $\lambda = (4, 4, 3, 1)$.



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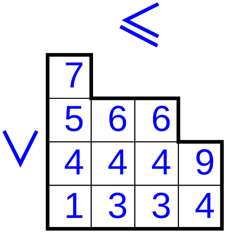
- Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431} = x_1^1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$$

Schur functions

Example

$$\leqslant \quad \vee \; \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array}$$

Hence

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &\quad +2x_1x_2x_3 \\ &= m_{21}(x_1, x_2, x_3) + 2m_{111}(x_1, x_2, x_3). \end{aligned}$$

Fact: Schur functions are symmetric functions.

Question

Why do we care about Schur functions?

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- Fact: The Schur functions form a basis for the symmetric functions.
- In fact, they form an orthonormal basis: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.
- Main reason: they arise in many other areas of mathematics. But first...

Note: The symmetric functions form a ring.

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

$c_{\mu\nu}^\lambda$: *Littlewood-Richardson coefficients*

s_λ and $c^\lambda_{\mu\nu}$ are superstars!

1. Representation Theory of $\mathcal{S}_n$:

$$(S^\mu \otimes S^\nu)\uparrow^{\mathcal{S}_n} = \bigoplus_\lambda c^\lambda_{\mu\nu} S^\lambda, \text{ so } \chi^\mu \cdot \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

   We also have that $s_\lambda =$ the Frobenius characteristic of χ^λ.

2. Representations of $\mathrm{GL}(n, \mathbb{C})$:
   $s_\lambda(x_1, \ldots, x_n) =$ the character of the irreducible rep. V^λ.

$$V^\mu \otimes V^\nu = \bigoplus c^\lambda_{\mu\nu} V^\lambda.$$

3. Algebraic Geometry: Schubert classes σ_λ form a linear basis for $H^*(\mathrm{Gr}_{kn})$. We have

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c^\lambda_{\mu\nu} \sigma_\lambda.$$

   Thus $c^\lambda_{\mu\nu}$ = number of points of Gr_{kn} in $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$.

There's more!

4. **Linear Algebra:** When do there exist Hermitian matrices *A*, *B* and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

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   When $c^{\lambda}_{\mu\nu} > 0$. (Heckman, Klyachko, Knutson, Tao.)

By 1, 2 or 3 we get:

$$c^{\lambda}_{\mu\nu} \geq 0.$$ (Your take-home fact!)

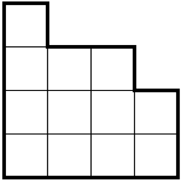
Consequences:

- We say that $s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda$ is a Schur-positive function.
- Want a combinatorial proof:
  "They must count something simpler!"

Skew Schur functions: a generalization of Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.

- Young diagram.
  Example:
  $\lambda = (4, 4, 3, 1)$



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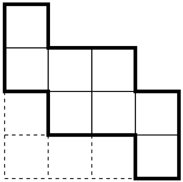

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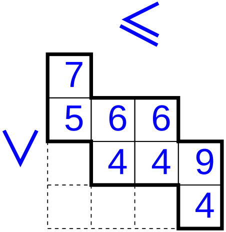

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$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \cdots$. Again, it's a symmetric function.
Remarkable fact:

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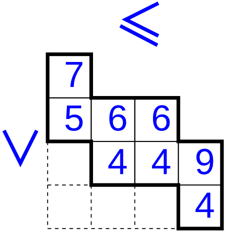

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Remarkable fact:

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_\nu .$$

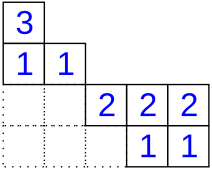
The Littlewood-Richardson rule

Littlewood-Richardson 1934, Schützenberger 1977, Glanffrwd Thomas 1974.

Theorem

$c^{\lambda}_{\mu\nu}$ equals the number of SSYT of shape λ/μ and content ν whose reverse reading word is a ballot sequence.

Example $\lambda = (5, 5, 2, 1), \mu = (3, 2), \nu = (4, 3, 1)$



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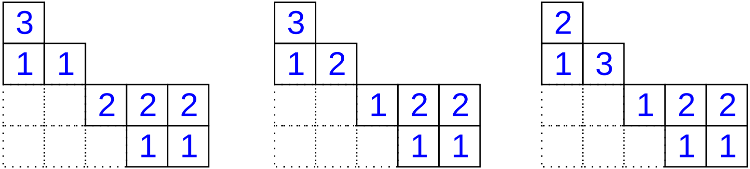

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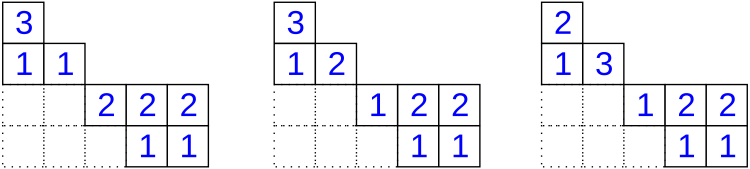

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$c^{5221}_{32,431} = 2.$ $c^{(12,11,10,9,8,7,6,5,4,3,2,1)}_{(8,7,6,5,4,3,2,1),(8,7,6,6,5,4,3,2,1)} = 7869992.$

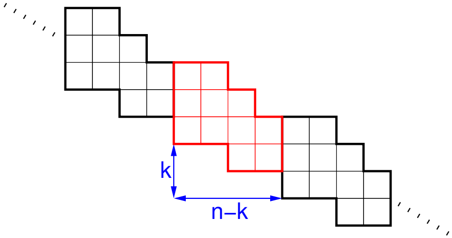
(Maple packages: John Stembridge, Anders Buch.)

The story so far

- Schur functions: (most?) important basis for the symmetric functions
- Skew Schur functions are Schur-positive
- The coefficients in the expansion are the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$
- $c^{\lambda}_{\mu\nu}$ = number of points of Gr_{kn} in $\tilde{\Omega}_{\mu} \cap \tilde{\Omega}_{\nu} \cap \tilde{\Omega}_{\lambda^{\vee}}$.
- The Littlewood-Richardson rule gives a combinatorial rule for calculating $c^{\lambda}_{\mu\nu}$, and hence much information about the other interpretations of $c^{\lambda}_{\mu\nu}$.

Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (a, b) and $(a + n - k, b - k)$.



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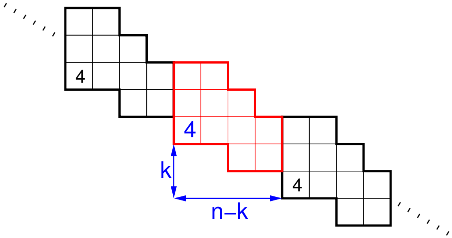

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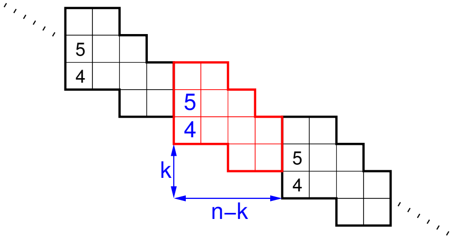

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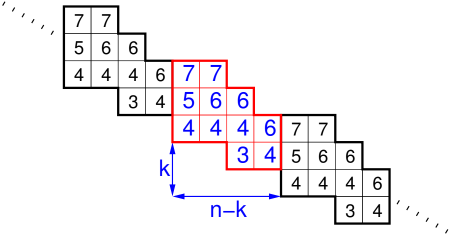

- Entries weakly increase in each row
  Strictly increase up each column
- Alternatively: SSYT with relations between entries in first and last columns
- Cylindric skew Schur function:

$$s_C(x) = \sum_T x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

$$\text{e.g. } s_C(x) = x_3 x_4^4 x_5 x_6^3 x_7^2 + \cdots .$$

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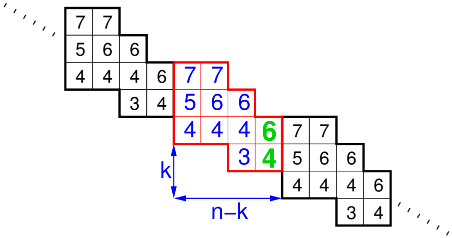

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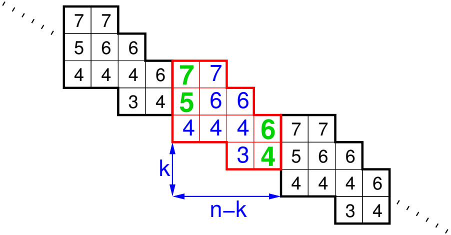

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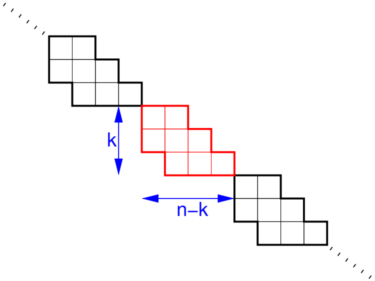
Skew shapes are cylindric skew shapes...

... and so skew Schur functions are cylindric skew Schur functions.

Example



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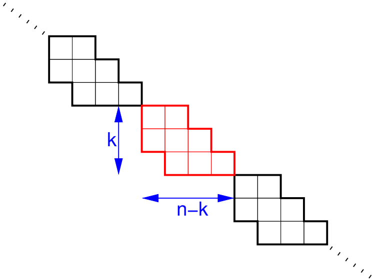

- Gessel, Krattenthaler: *"Cylindric partitions,"* 1997.
- Bertram, Ciocan-Fontanine, Fulton: *"Quantum multiplication of Schur polynomials,"* 1999.
- Postnikov: *"Affine approach to quantum Schubert calculus,"* `math.CO/0205165`.
- Stanley: *"Recent developments in algebraic combinatorics,"* `math.CO/0211114`.

Motivation: Positivity of Gromov-Witten invariants

In $H^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu \sigma_\nu = \sum_\lambda c_{\mu\nu}^\lambda \sigma_\lambda.$$

In $QH^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\lambda \subseteq k \times (n-k)} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda.$$

$C_{\mu\nu}^{\lambda,d}$ = 3-point Gromov-Witten invariants
= $\#\{$rational curves of degree d in Gr_{kn} that meet $\tilde{\Omega}_\mu$, $\tilde{\Omega}_\nu$ and $\tilde{\Omega}_{\lambda^\vee}\}$.

Example

$$C_{\mu,\nu}^{\lambda,0} = c_{\mu\nu}^\lambda.$$

Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.

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Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.
"Fundamental open problem": Find an algebraic or combinatorial proof of this fact.

Connection with cylindric skew Schur functions

Theorem (Postnikov)

$$s_{\mu/d/\nu}(x_1, \ldots, x_k) = \sum_{\lambda \subseteq k \times (n-k)} C_{\mu\nu}^{\lambda,d} s_\lambda(x_1, \ldots, x_k).$$

Conclusion: Want to understand the expansions of cylindric skew Schur functions into Schur functions.

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Corollary

$s_{\mu/d/\nu}(x_1, \ldots, x_k)$ is Schur-positive.

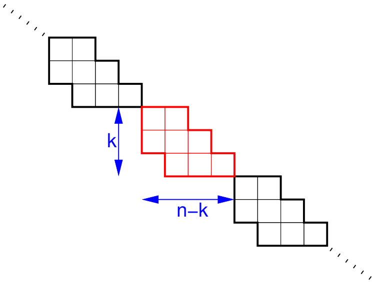
Known: $s_{\mu/d/\nu}(x_1, x_2, \ldots) \equiv s_{\mu/d/\nu}(x)$ need not be Schur-positive.

Example

If $s_{\mu/d/\nu} = s_{22} + s_{211} - s_{1111}$, then $s_{\mu/d/\nu}(x_1, x_2, x_3)$ is Schur-positive.

(In general: $s_\lambda(x_1, \ldots, x_k) \neq 0 \Leftrightarrow \lambda$ has at most k parts.)

When is a cylindric skew Schur function Schur-positive?



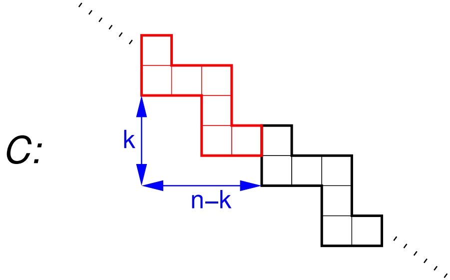
Theorem (McN.)

For any cylindric skew shape C,

$$s_C(x_1, x_2, \ldots) \textit{ is Schur-positive} \Leftrightarrow C \textit{ is a skew shape}.$$

Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \ldots)$ is not Schur-positive.

Example: cylindric ribbons



$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})}$$
$$+ s_{(n-k-2,1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

Formula: cylindric skew Schur functions as signed sums of skew Schur functions

Idea for formulation: Bertram, Ciocan-Fontanine, Fulton
Uses result of Gessel, Krattenthaler

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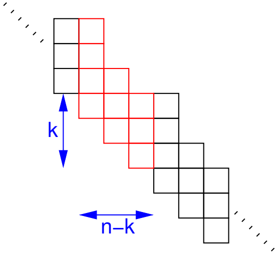

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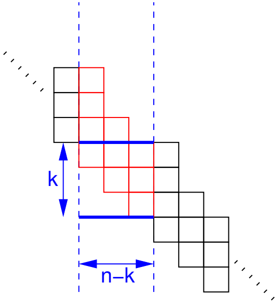

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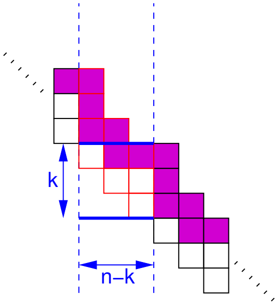

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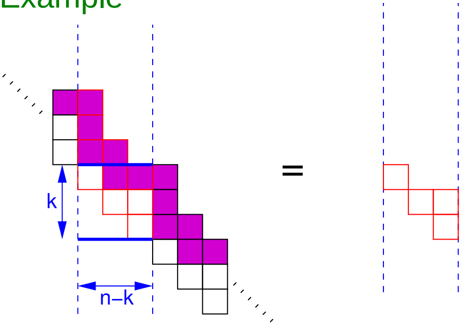

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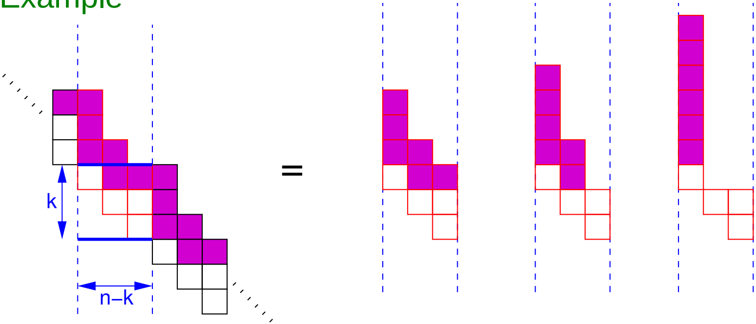

Formula: cylindric skew Schur functions as signed sums of skew Schur functions

Idea for formulation: Bertram, Ciocan-Fontanine, Fulton
Uses result of Gessel, Krattenthaler

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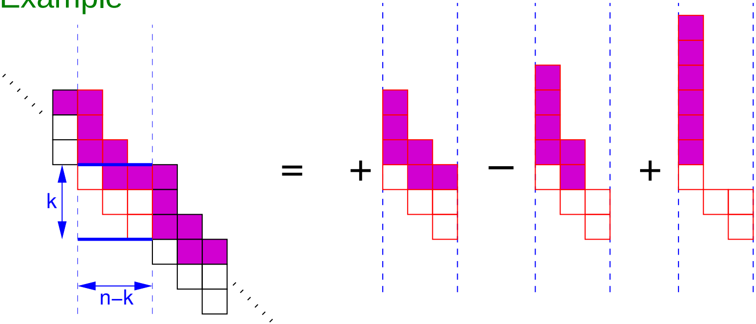

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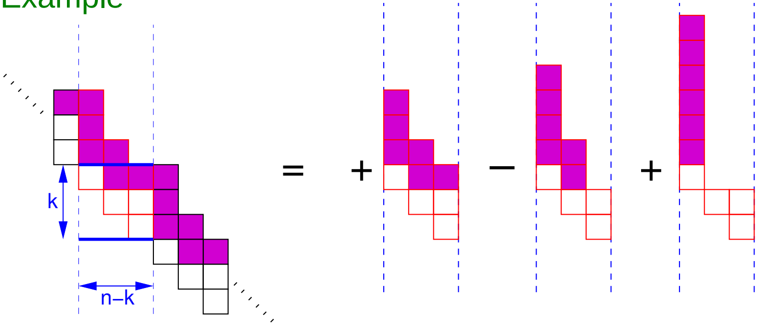
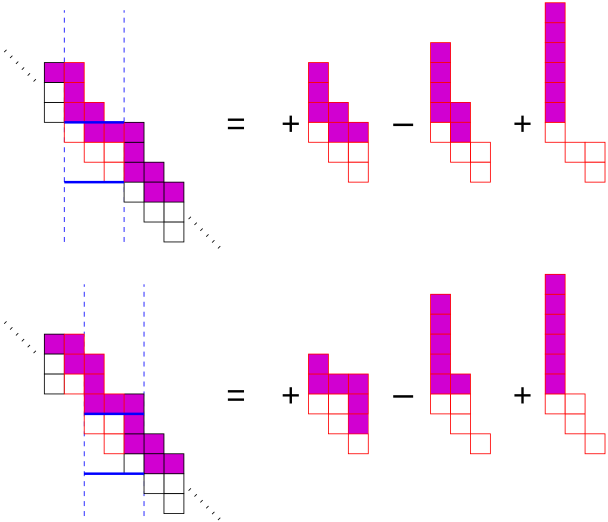
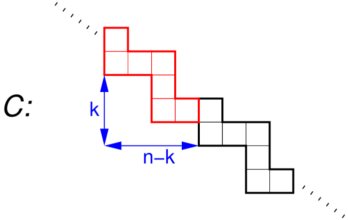


$$s_C = s_{333211/21} - s_{3322111/21} + s_{331111111/21}$$
$$= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111}$$
$$- s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.$$

First consequence: lots of skew Schur function identities



A final thought: shouldn't cylindric skew Schur functions be Schur-positive *in some sense*?



$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})}$$
$$+ s_{(n-k-2,1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

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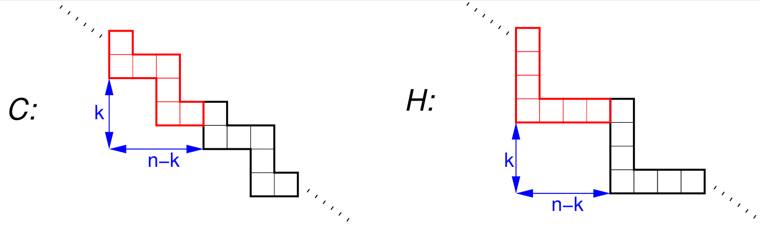

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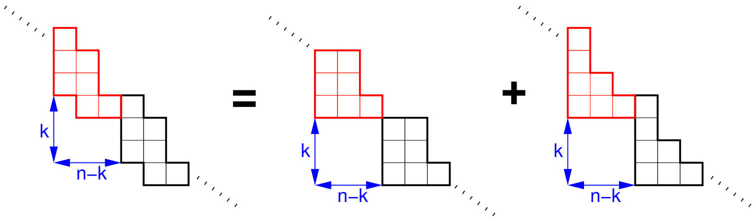
In fact,

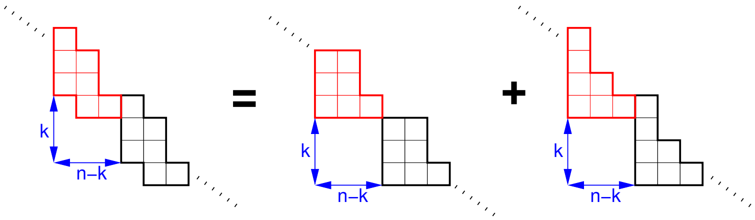
$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_H.$$

s_C: cylindric *skew* Schur function
s_H: cylindric Schur function
We say that s_C is cylindric Schur-positive.

A Conjecture



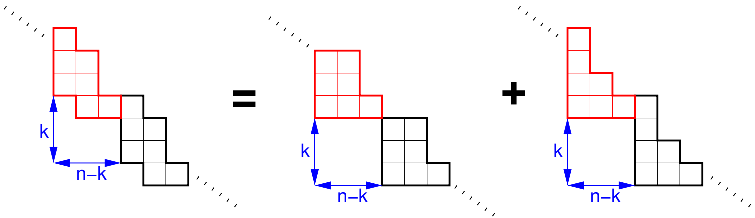
A Conjecture



Conjecture

For any cylindric skew shape C, s_C is cylindric Schur-positive

A Conjecture



Conjecture

For any cylindric skew shape C, s_C is cylindric Schur-positive

Theorem (McN.)

The cylindric Schur functions corresponding to a given translation $(-n+k,+k)$ are linearly independent.

Theorem (McN.)

If s_C can be written as a linear combination of cylindric Schur functions with the same translation as C, then s_C is cylindric Schur-positive.

Summary of results

- Classification of those cylindric skew Schur functions that are Schur-positive.
- Full knowledge of negative terms in Schur expansion of ribbons.
- Expansion of any cylindric skew Schur function into skew Schur functions.
- Conjecture and evidence that every cylindric skew Schur function is cylindric Schur-positive.

- Outlook
  - Prove the conjecture
  - Develop a Littlewood-Richardson rule for cylindric skew Schur functions - this would solve the “fundamental open problem.”

Another Schur-positivity research project

Know

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$$

is Schur-positive.

Question

Given μ, ν, when is

$$s_\sigma s_\tau - s_\mu s_\nu$$

Schur-positive? In other words, when is $c_{\sigma\tau}^\lambda - c_{\mu\nu}^\lambda \geq 0$ for every partition λ.

- Fomin, Fulton, Li, Poon: *"Eigenvalues, singular values, and Littlewood-Richardson coefficients,"* `math.AG/0301307`.
- Bergeron, Biagioli, Rosas: *"Inequalities between Littlewood-Richardson Coefficients,"* `math.CO/0403541`.
- Bergeron, McNamara: *"Some positive differences of products of Schur functions,"* `math.CO/0412289`.

Proof. Consider SSYTs of shape λ and *content* $\alpha = (\alpha_1, \alpha_2, \ldots)$.

Show: $\#$ SSYTs shape λ, content $\alpha = \#$ SSYTs shape λ, content β, where β is any permutation of α.

Sufficient: $\beta = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots)$.

Bijection: SSYTs shape λ, content $\alpha \leftrightarrow$ SSYTs shape λ, content β.

$$\begin{array}{ccccccccccc} i+1 & i+1 & & & & & & & \\ i & i & \underbrace{i \quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\ & & & & i \end{array}$$

In each such row, convert the r i's and s $i+1$'s to s i's and r $i+1$'s:

$$\begin{array}{ccccc} i+1 & i+1 & & & \\ i & i & \underbrace{i \quad i \quad i \quad i}_{s=4} & \underbrace{i+1 \quad i+1}_{r=2} & i+1 \\ & & & & i \end{array}$$

$\square$