# Conjectures concerning the difference of two skew Schur functions

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*Positivity in Algebraic Combinatorics*
Banff International Research Station

15 August 2015

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`www.facstaff.bucknell.edu/pm040/`

# The setting

$s_A$: the skew Schur function for the skew shape $A$

Overarching Question. For skew shapes $A$ and $B$, when is

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Schur-positive?
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Special Case. For partitions $\alpha, \beta, \gamma, \delta$, when is

$$s_\alpha s_\beta - s_\gamma s_\delta$$

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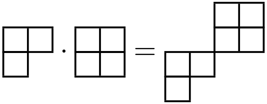

[Azenhas, Ballantine, F. Bergeron, Biagioli, Conflitti, Fomin, Fulton, King, A. N. Kirillov, Lam, Lascoux, Leclerc, C.-K. Li, Mamede, M., Okounkov, Orellana, Poon, Postnikov, Pylyavskyy, Rosas, Thibon, Welsh, van Willigenburg, ...]

# The problems and conjectures

1. Equality of skew Schur functions
   *Joint with Stephanie van Willigenburg*
2. Connected skew Schur functions maximal in Schur-positivity order
   *Joint with Pavlo Pylyavskyy and Stephanie van Willigenburg*
3. *F*-support containment and the row-overlap conditions of Reiner, Shaw and van Willigenburg
4. A Saturation Theorem for skew Schur functions
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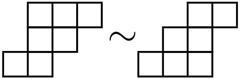
Determine necessary and sufficient conditions on shapes of $A$ and $B$.



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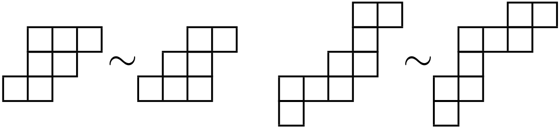

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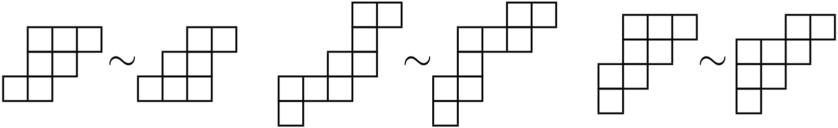

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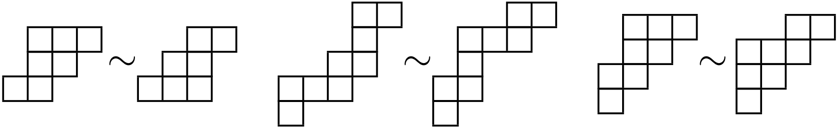

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- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006): 3 operations for generating skew shapes with equal skew Schur functions; necessary conditions
- M., Steph van Willigenburg (2006): unification, generalization, conjecture for necessary and sufficient conditions
- Christian Gutschwager (2008): multiplicity-free skew shapes

# 1. Equality of skew Schur functions

(With apologies)

Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].

Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r \text{ , where} \end{aligned}$$

- ○ $E_i = W_i O_i W_i$ satisfies four hypotheses for all $i$,
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Evidence [M., van Willigenburg, (2006)].

- With one more hypothesis, the “if” direction
  Proof uses results of Hamel–Goulden and Chen–Yan–Yang.
- $n \leq 20$

Evidence [Gutschwager, 2006)]. Multiplicity-free skew shapes

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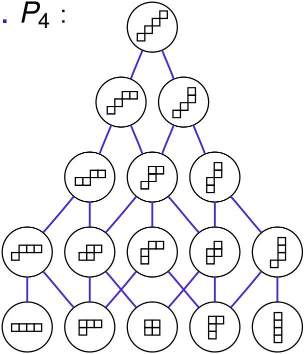
**Definition.** Let $A$, $B$ be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

If $B \leq_s A$ then $|A| = |B|$.

**Example.** $P_4$ :



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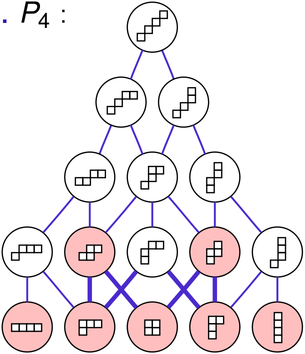

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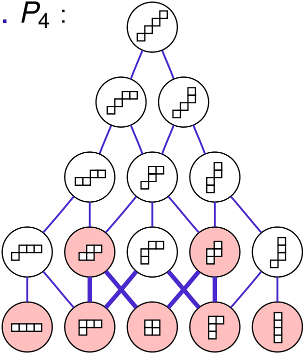
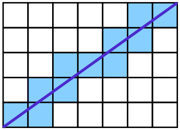


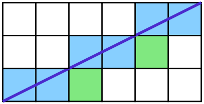
**Problem 2.**
What are the maximal elements of $P_n$ among the connected skew shapes?

# 2. Maximal connected skew shapes

Conjecture 2 [M., Pylyavskyy (2007)]. For each $r = 1, \ldots, n$, there is a unique maximal connected element with $r$ rows, namely the ribbon marked out by the diagonal of an $r$-by-$(n-r+1)$ box.

Examples.





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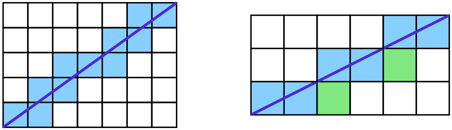

Evidence [M., van Willigenburg (2011)].

- $n \leq 34$
- Maximal element must be an equitable ribbon: row (resp. column) lengths differ by at most 1.
- $\mathrm{Supp}_s(A) := \{\lambda \vdash n \mid s_\lambda$ appears in the Schur expansion of $s_A\}$, the Schur-support of $A$.

  e.g. $s$ $= s_3 + 2s_{21} + s_{111}$. $\mathrm{Supp}_s($$) = \{3, 21, 111\}$.

  True in Support Poset: $A \geq_{\mathrm{Supp}_s} B$ if $\mathrm{Supp}_s(A) \supseteq \mathrm{Supp}_s(B)$.

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**Definition [Reiner, Shaw, van Willigenburg].** For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.
Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

**Example.**

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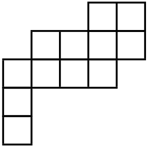

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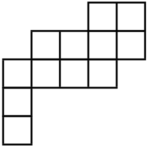

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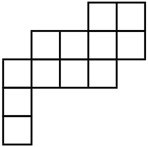

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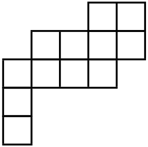
$A =$ 

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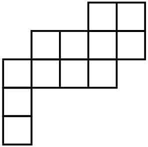

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- $\mathrm{rows}_k(A) = \emptyset$ for $k > 3$.

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Necessary conditons for equality

Theorem [RSvW, (2006)]. Let $A$ and $B$ be skew shapes. If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \text{ for all } k.$$

Question. What are necessary conditions on $A$ and $B$ for $s_A - s_B$ to be Schur-positive?

Theorem [M., (2008)]. Let $A$ and $B$ be skew shapes. If $s_A - s_B$ is Schur-positive, then

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In fact, it suffices to assume that $\mathrm{Supp}_s(A) \supseteq \mathrm{Supp}_s(B)$.

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**Theorem [M., (2008)].**

$\boxed{s_A - s_B \text{ is Schur-pos.}} \quad \Rightarrow \quad \boxed{\mathrm{Supp}_s(A) \supseteq \mathrm{Supp}_s(B)}$

$\Searrow$

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Equivalent choices:

$\mathrm{cols}_\ell(A) \preceq_{\mathrm{dom}} \mathrm{cols}_\ell(B)\ \forall \ell$

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Converse is already false at $n = 4$.

Problem 3. What weaker algebraic conditions best fill the gap?

# 3. The row-overlap conditions and $F$-support containment

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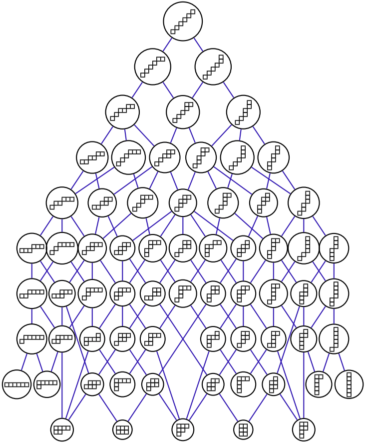
**Conjecture 4 [M., (2013)].** The rightmost implication is if and only if.

**Evidence [M., (2013)].** Conjecture is true for:

- $n \leq 13$ (compare with failure at $n = 4$ for other converse implications);
- $F$-multiplicity-free skew shapes (as classified by Christine Bessenrodt and Steph van Willigenburg, (2013));
- ribbons whose rows all have length at least 2.

# 3. The row-overlap conditions and $F$-support containment

Example. $n = 6$



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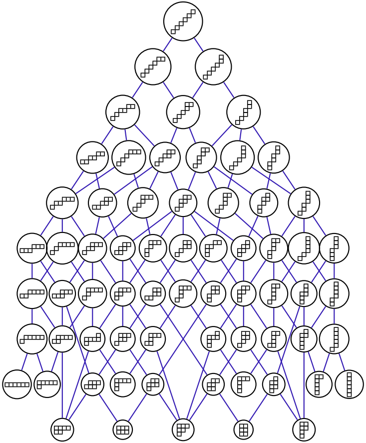

Dual of row overlap dominance

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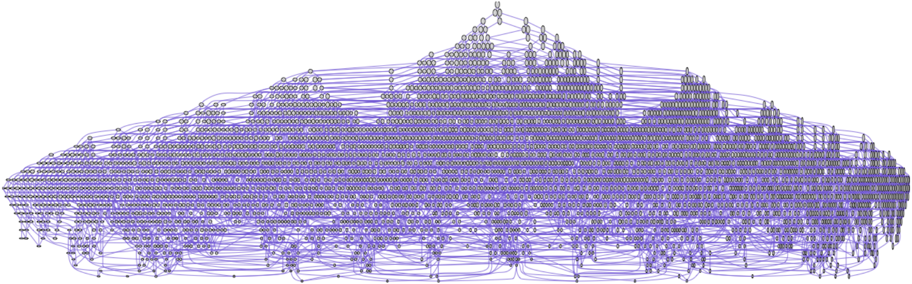
Example. $n = 12$ case has 12,042 edges.



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Answer. No. False even in the easier direction $\Rightarrow$.

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$$\nu \in \mathrm{Supp}_s(A) \iff n\nu \in \mathrm{Supp}_s(nA).$$

Equivalently,

$$\mathrm{Supp}_s(\nu) \subseteq \mathrm{Supp}_s(A) \iff \mathrm{Supp}_s(n\nu) \subseteq \mathrm{Supp}_s(nA).$$

Problem 4. [Speyer (2009)]. Can this be generalized by replacing $\nu$ by a skew shape?

Answer. No. False even in the easier direction ⇒.

Conjecture 4 [M., Morales (2014)]. A quasisymmetric skew Saturation Theorem:

$$\mathrm{Supp}_F(B) \subseteq \mathrm{Supp}_F(A) \iff \mathrm{Supp}_F(nB) \subseteq \mathrm{Supp}_F(nA).$$

# 4. A Saturation Theorem for skew Schur functions

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Evidence. Follows from Conjecture 3.

# 1. Equality of skew Schur functions

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Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].

Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r \ , \text{ where} \end{aligned}$$

∘ $E_i = W_i O_i W_i$ satisfies four hypotheses for all $i$,
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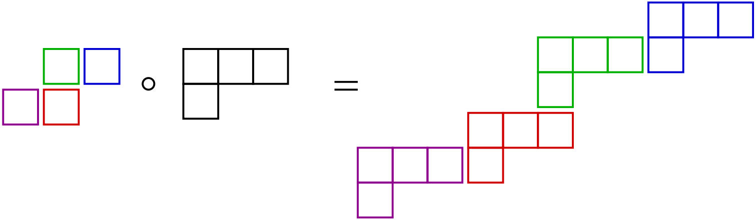
# 1. Equality of skew Schur functions

## Composition of skew shapes

$D \circ E =$



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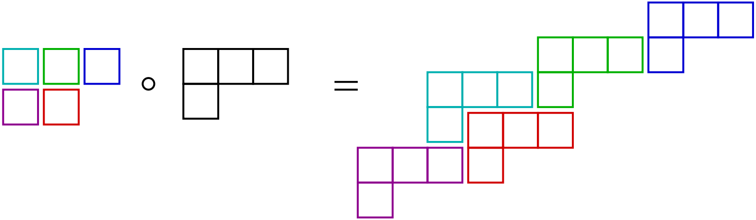

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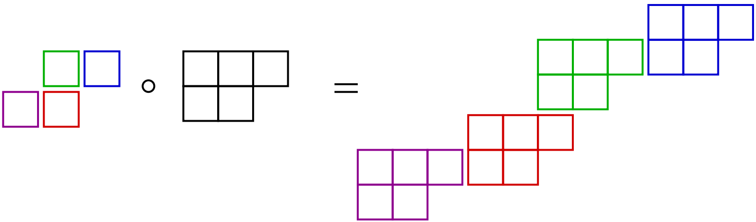

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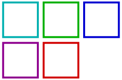
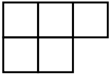
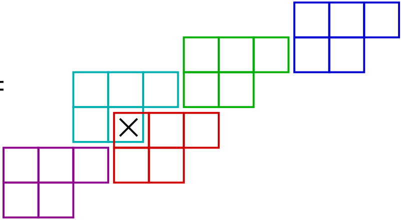


$\circ$



$=$



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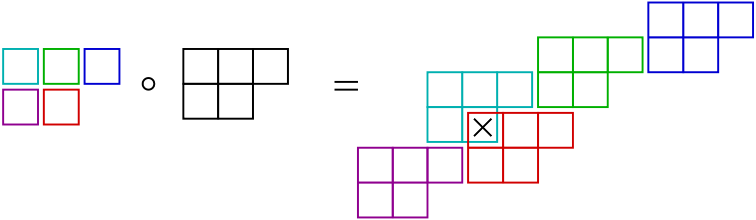
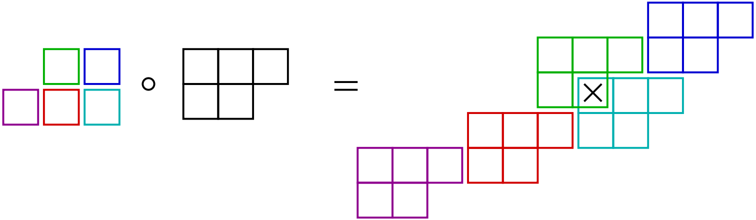
$D \circ E =$ 

Theorem [M., van Willigenburg, (2006)]. If $D \sim D'$, then

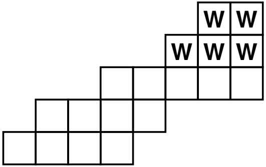
$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$



# 1. Equality of skew Schur functions

Amalgamated compositions: $\circ_W$

A skew shape $W$ lies in the top of a skew shape $E$ if $W$ appears as a connected subshape of $E$ that includes the northeasternmost cell of $E$.



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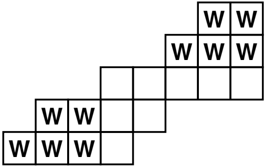

Similarly, $W$ lies in the bottom of $E$.

Our interest. $W$ lies in both the top and bottom of $E$. We write $E = W O W$.

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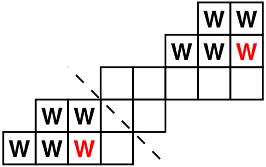

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Our interest. $W$ lies in both the top and bottom of $E$. We write $E = W O W$.

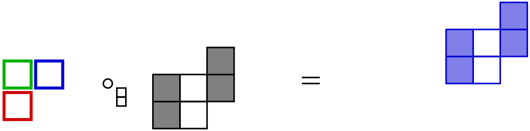
Hypotheses [inspired by hypotheses of RSvW].

1. $W_{ne}$ and $W_{sw}$ are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew shapes.
3. $W$ is maximal given its set of diagonals.

# 1. Equality of skew Schur functions

Example.

$D \circ_W E =$



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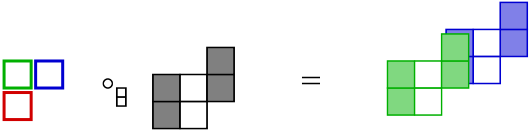

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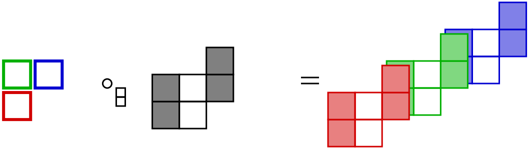

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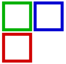
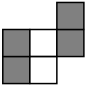
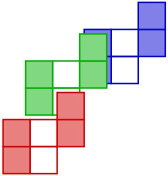
$D \circ_W E =$  $\circ$  $=$ 

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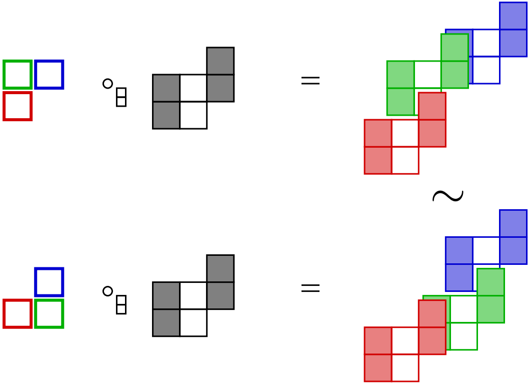
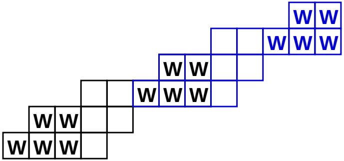


# 1. Equality of skew Schur functions

Construction of $\overline{W}$ and $\overline{O}$:



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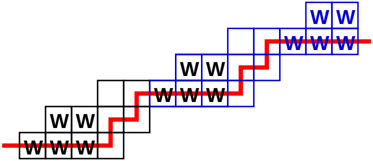

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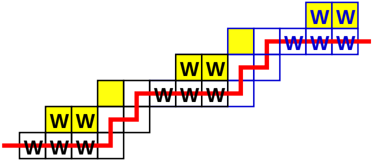

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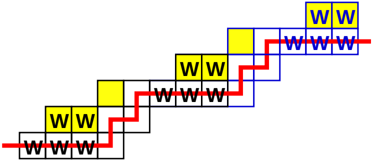

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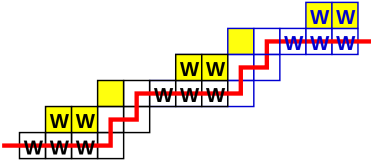

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**Thanks!**