

Conjectures concerning the difference of two skew Schur functions

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Positivity in Algebraic Combinatorics
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www.facstaff.bucknell.edu/pm040/

The setting

s_A : the skew Schur function for the skew shape A

Overarching Question. For skew shapes A and B , when is

$$s_A - s_B$$

Schur-positive?

Want simple conditions in terms of the shapes of A and B .

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Special Case. For partitions $\alpha, \beta, \gamma, \delta$, when is

$$s_\alpha s_\beta - s_\gamma s_\delta$$

Schur-positive?

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

[Azenhas, Ballantine, F. Bergeron, Biagioli, Conflitti, Fomin, Fulton, King, A. N. Kirillov, Lam, Lascoux, Leclerc, C.-K. Li, Mamede, M., Okounkov, Orellana, Poon, Postnikov, Pylyavskyy, Rosas, Thibon, Welsh, van Willigenburg, ...]

The problems and conjectures

1. Equality of skew Schur functions
Joint with Stephanie van Willigenburg
2. Connected skew Schur functions maximal in Schur-positivity order
Joint with Pavlo Pylyavskyy and Stephanie van Willigenburg
3. F -support containment and the row-overlap conditions of Reiner, Shaw and van Willigenburg
4. A Saturation Theorem for skew Schur functions
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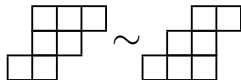


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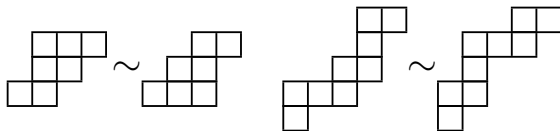
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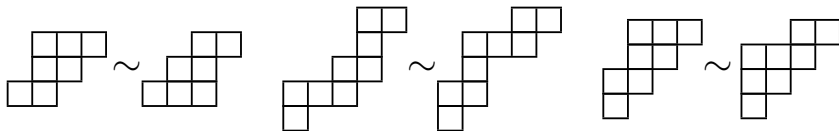


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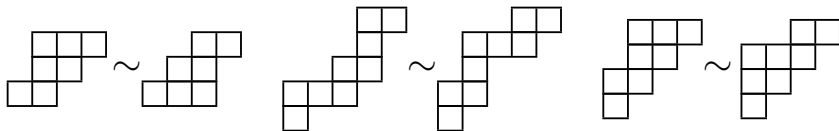


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- ▶ John Stembridge (2004): skewed staircases
- ▶ Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006): 3 operations for generating skew shapes with equal skew Schur functions; necessary conditions
- ▶ M., Steph van Willigenburg (2006): unification, generalization, conjecture for necessary and sufficient conditions
- ▶ Christian Gutschwager (2008): multiplicity-free skew shapes

1. Equality of skew Schur functions

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Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].

Two skew shapes E and E' satisfy $E \sim E'$ if and only if, for some r ,

$$\begin{aligned} E &= ((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r, \text{ where} \end{aligned}$$

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Evidence [M., van Willigenburg, (2006)].

- ▶ With one more hypothesis, the “if” direction
Proof uses results of Hamel–Goulden and Chen–Yan–Yang.
- ▶ $n \leq 20$

Evidence [Gutschwager, 2006]. Multiplicity-free skew shapes

2. Maximal connected skew shapes

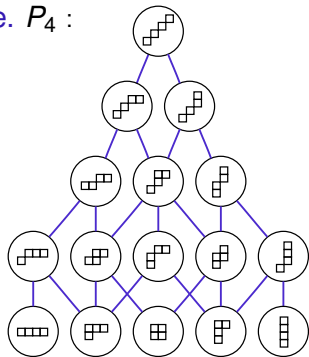
2. Maximal connected skew shapes

Definition. Let A, B be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

If $B \leq_s A$ then $|A| = |B|$.

Example. P_4 :



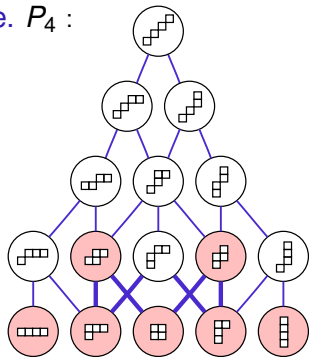
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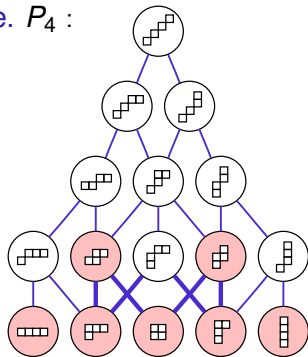
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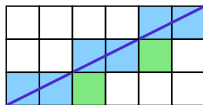
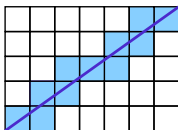
Problem 2.

What are the maximal elements of P_n among the **connected** skew shapes?

2. Maximal connected skew shapes

Conjecture 2 [M., Pylyavskyy (2007)]. For each $r = 1, \dots, n$, there is a unique maximal connected element with r rows, namely the ribbon marked out by the diagonal of an r -by- $(n - r + 1)$ box.

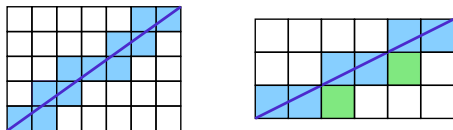
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Examples.



Evidence [M., van Willigenburg (2011)].

- ▶ $n \leq 34$
- ▶ Maximal element must be an **equitable** ribbon: row (resp. column) lengths differ by at most 1.
- ▶ $\text{Supp}_s(A) := \{\lambda \vdash n \mid s_\lambda \text{ appears in the Schur expansion of } s_A\}$, the **Schur-support** of A .

e.g. $s_{\square^4} = s_3 + 2s_{21} + s_{111}$. $\text{Supp}_s(\square^4) = \{3, 21, 111\}$.

True in **Support Poset**: $A \geq_{\text{Supp}_s} B$ if $\text{Supp}_s(A) \supseteq \text{Supp}_s(B)$.

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General idea: the overlaps among rows must match up for $s_A = s_B$.

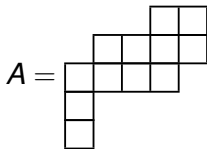
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Definition [Reiner, Shaw, van Willigenburg]. For a skew shape A , let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \dots, i+k-1$.

Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \dots)$.

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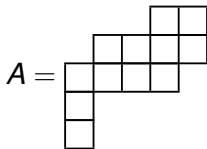
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- ▶ $\text{overlap}_1(i) = \text{length of the } i\text{th row}$. Thus $\text{rows}_1(A) = 44211$.

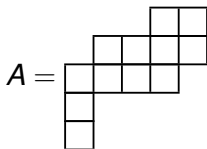
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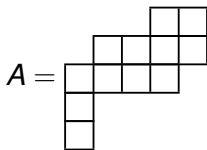
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- ▶ $\text{rows}_3(A) = 11$.

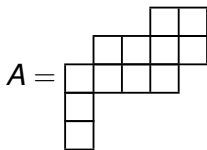
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- ▶ $\text{rows}_k(A) = \emptyset$ for $k > 3$.

3. The row-overlap conditions

Necessary conditions for equality

Theorem [RSvW, (2006)]. Let A and B be skew shapes.

If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \text{ for all } k.$$

Question. What are necessary conditions on A and B for $s_A - s_B$ to be Schur-positive?

Theorem [M., (2008)]. Let A and B be skew shapes. If $s_A - s_B$ is Schur-positive, then

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In fact, it suffices to assume that $\text{Supp}_s(A) \supseteq \text{Supp}_s(B)$.

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\Rightarrow

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Equivalent choices:

$\text{cols}_\ell(A) \leq_{\text{dom}} \text{cols}_\ell(B) \forall \ell$

$\text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \forall k, \ell$

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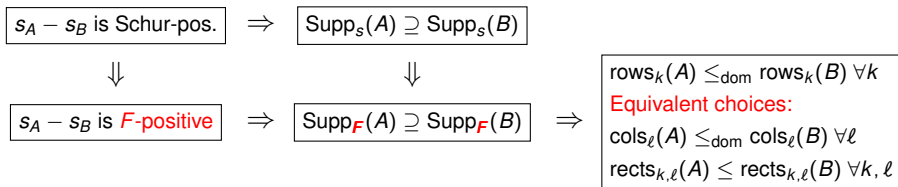
$$\begin{aligned} \text{rows}_k(A) &\leq_{\text{dom}} \text{rows}_k(B) \quad \forall k \\ \text{Equivalent choices:} \\ \text{cols}_\ell(A) &\leq_{\text{dom}} \text{cols}_\ell(B) \quad \forall \ell \\ \text{rects}_{k,\ell}(A) &\leq \text{rects}_{k,\ell}(B) \quad \forall k, \ell \end{aligned}$$

Converse is already false at $n = 4$.

Problem 3. What weaker algebraic conditions best fill the gap?

3. The row-overlap conditions and F -support containment

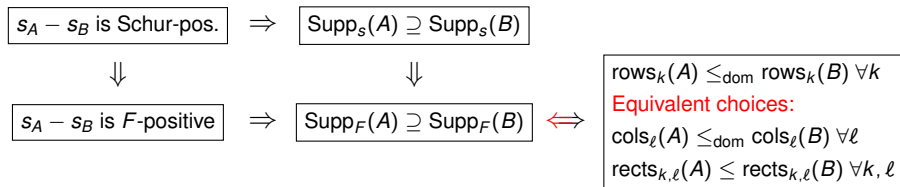
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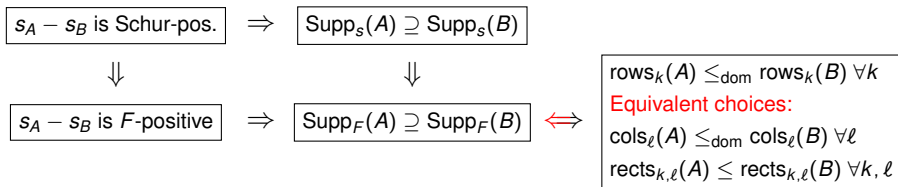


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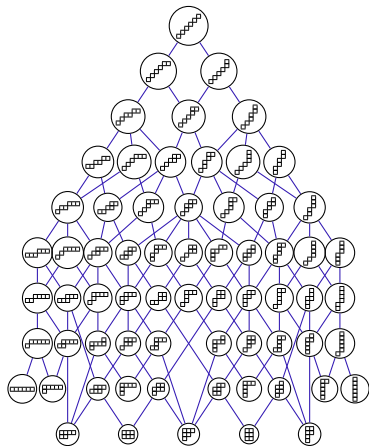
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Evidence [M., (2013)]. Conjecture is true for:

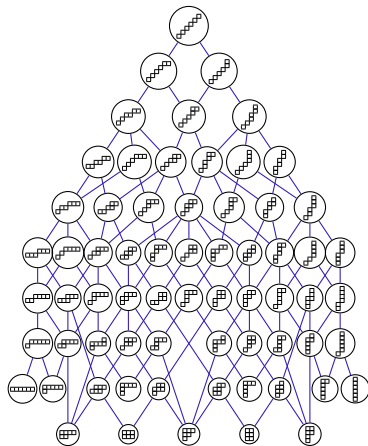
- ▶ $n \leq 13$ (compare with failure at $n = 4$ for other converse implications);
- ▶ F -multiplicity-free skew shapes (as classified by [Christine Bessenrodt and Steph van Willigenburg, \(2013\)](#));
- ▶ ribbons whose rows all have length at least 2.

3. The row-overlap conditions and F -support containment

Example. $n = 6$



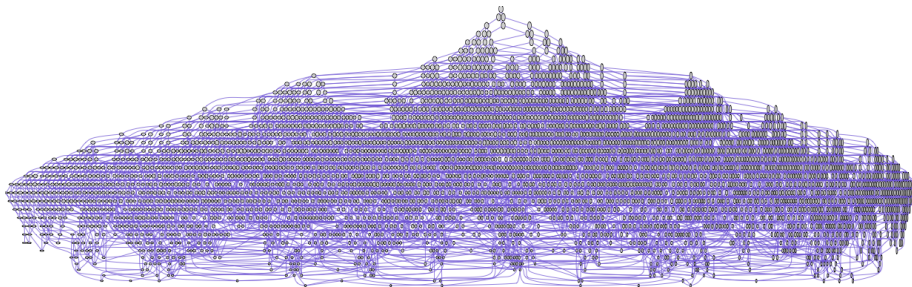
F -support containment



Dual of row overlap dominance

3. The row-overlap conditions and F -support containment

Example. $n = 12$ case has 12,042 edges.



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$A = \lambda/\mu$ and k is a positive integer, define $kA = k\lambda/k\mu$.

Theorem [Knutson, Tao, (1999)]. For a skew shape A and partition ν ,

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Conjecture 4 [M., Morales (2014)]. A quasisymmetric skew Saturation Theorem:

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Evidence. Follows from Conjecture 3.

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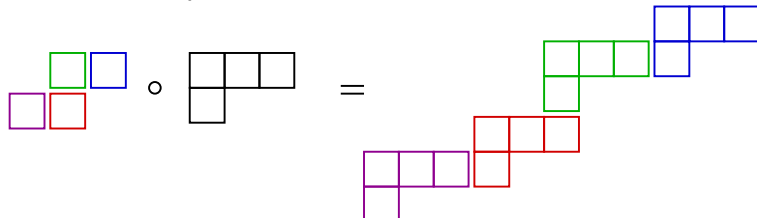
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Composition of skew shapes

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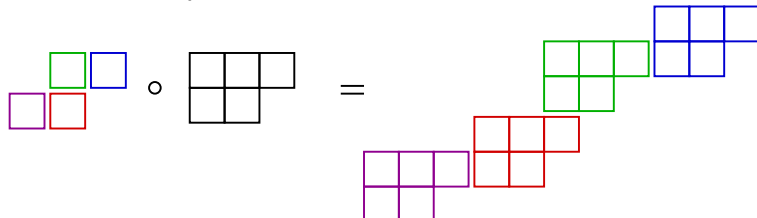
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Composition of skew shapes

$$D \circ E =$$


The diagram illustrates the composition of two skew shapes. On the left, a purple square (1x1) is below a red square (1x1), which is below a green square (1x1) and a blue square (1x1). This is followed by a circle symbol and a black skew shape consisting of a top row of three squares and a bottom row of two squares. An equals sign follows, leading to a larger skew shape composed of four colored components: a purple component (bottom-left, 2x2 with a 1x1 extension), a red component (middle-left, 2x2 with a 1x1 extension), a green component (top-middle, 2x2 with a 1x1 extension), and a blue component (top-right, 2x2 with a 1x1 extension).

1. Equality of skew Schur functions

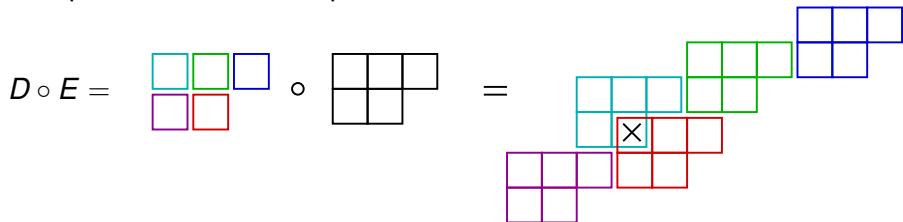
Composition of skew shapes

$$D \circ E = \begin{array}{|c|c|c|} \hline \color{cyan}\square & \color{green}\square & \color{blue}\square \\ \hline \color{purple}\square & \color{red}\square & \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & & & \color{blue}\square & \color{blue}\square & \color{blue}\square \\ \hline & & & & \color{blue}\square & & \\ \hline & & & & \color{green}\square & \color{green}\square & \color{green}\square & \color{green}\square \\ \hline & & & & \color{green}\square & & \\ \hline & & & & \color{cyan}\square & \color{cyan}\square & \color{cyan}\square & \color{cyan}\square \\ \hline & & & & \color{cyan}\square & & \\ \hline & & & & \color{red}\square & \color{red}\square & \color{red}\square & \color{red}\square \\ \hline & & & & \color{red}\square & & \\ \hline & & & & \color{purple}\square & \color{purple}\square & \color{purple}\square & \color{purple}\square \\ \hline & & & & \color{purple}\square & & \\ \hline \end{array}$$

The diagram illustrates the composition of two skew shapes, D and E , resulting in a single skew shape. The shape D is a 2x3 grid with colored cells: cyan (top-left), green (top-middle), blue (top-right), purple (bottom-left), and red (bottom-middle). The shape E is a 2x3 grid with black cells: top-left, top-middle, top-right, bottom-left, and bottom-middle. The resulting shape is a 6x8 grid where the cells are colored according to the composition: purple (bottom-left 2x2), red (bottom-middle 2x2), cyan (bottom-right 2x2), green (top-middle 2x2), and blue (top-right 2x2). A black 'X' is placed in the top-left cell of the red 2x2 block.

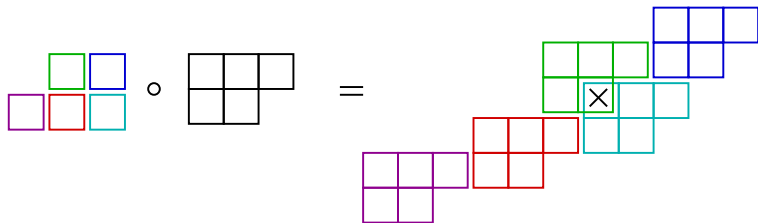
1. Equality of skew Schur functions

Composition of skew shapes



Theorem [M., van Willigenburg, (2006)]. If $D \sim D'$, then

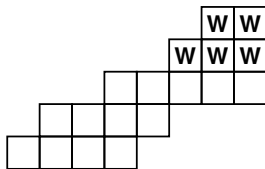
$$D' \circ E \sim D \circ E \sim D \circ E^*.$$



1. Equality of skew Schur functions

Amalgamated compositions: \circ_W

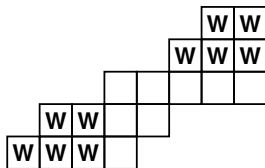
A skew shape W lies in the top of a skew shape E if W appears as a connected subshape of E that includes the northeasternmost cell of E .



1. Equality of skew Schur functions

Amalgamated compositions: \circ_W

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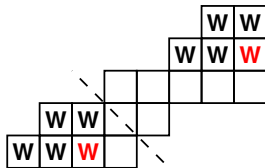
Similarly, W **lies in the bottom** of E .

Our interest. W lies in both the top and bottom of E . We write $E = WOW$.

1. Equality of skew Schur functions

Amalgamated compositions: \circ_W

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Hypotheses [inspired by hypotheses of RSvW].

1. W_{ne} and W_{sw} are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew shapes.
3. W is maximal given its set of diagonals.


1. Equality of skew Schur functions

Example.

$$D \circ_W E = \begin{array}{|c|c|} \hline \color{green}\square & \color{blue}\square \\ \hline \color{red}\square & \\ \hline \end{array} \circ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \color{blue}\square & \color{blue}\square \\ \hline \color{blue}\square & \color{blue}\square \\ \hline \end{array}$$

1. Equality of skew Schur functions

Example.

$$D \circ_W E =$$


The diagram illustrates the composition of two skew Schur functions, D and E , resulting in a new skew Schur function. The function D is represented by a 2x2 grid of squares: the top-left square is green, the top-right is blue, and the bottom-left is red. The function E is represented by a 2x2 grid of squares: the top-left is white, the top-right is gray, the bottom-left is gray, and the bottom-right is white. The result of the composition $D \circ_W E$ is a skew Schur function represented by a 2x2 grid of squares: the top-left square is green, the top-right is white, the bottom-left is white, and the bottom-right is blue. The green and blue squares are overlaid on the white squares.

1. Equality of skew Schur functions

Example.

$$D \circ_W E = \begin{array}{|c|c|} \hline \color{green}\square & \color{blue}\square \\ \hline \color{red}\square & \\ \hline \end{array} \circ_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} \begin{array}{|c|c|c|} \hline \color{gray}\square & \square & \color{gray}\square \\ \hline \color{gray}\square & \square & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \color{red}\square & \square & \color{red}\square & \color{green}\square \\ \hline \color{red}\square & \square & \color{green}\square & \color{blue}\square \\ \hline \end{array}$$

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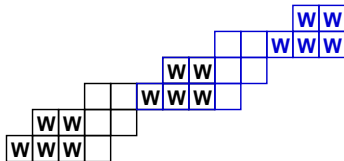
1. Equality of skew Schur functions

Example.

$$\begin{array}{c}
 D \circ_W E = \\
 \begin{array}{ccc}
 \begin{array}{|c|c|} \hline \color{green}{\square} & \color{blue}{\square} \\ \hline \color{red}{\square} & \\ \hline \end{array} & \circ_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} & \begin{array}{|c|c|c|} \hline & & \color{gray}{\square} \\ \hline \color{gray}{\square} & \square & \color{gray}{\square} \\ \hline \color{gray}{\square} & \square & \\ \hline \end{array} & = & \begin{array}{|c|c|c|c|} \hline & & & \color{blue}{\square} \\ \hline & & \color{green}{\square} & \color{blue}{\square} \\ \hline & \color{green}{\square} & & \\ \hline \color{red}{\square} & \color{red}{\square} & & \\ \hline \color{red}{\square} & & & \\ \hline \end{array} \\
 \\
 \begin{array}{ccc}
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 \end{array} \sim \begin{array}{|c|c|c|c|} \hline & & & \color{blue}{\square} \\ \hline & & \color{blue}{\square} & \color{blue}{\square} \\ \hline & & \color{green}{\square} & \\ \hline \color{red}{\square} & \color{red}{\square} & & \\ \hline \color{red}{\square} & & & \\ \hline \end{array}
 \end{array}$$

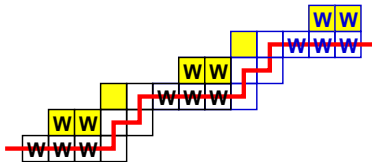
1. Equality of skew Schur functions

Construction of \overline{W} and \overline{O} :



1. Equality of skew Schur functions

Construction of \overline{W} and \overline{O} :



Hypothesis 4. \overline{W} is never adjacent to \overline{O} .

Conjecture 1.

Two skew shapes E and E' satisfy $E \sim E'$ if and only if, for some r ,

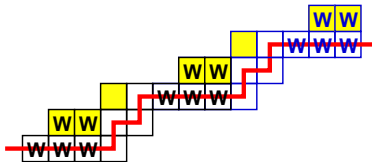
$$E = ((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r$$

$$E' = ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r, \text{ where}$$

- $E_i = W_i O_i W_i$ satisfies Hypotheses 1–4 for all i ,
- E'_i and W'_i denote either E_i and W_i , or E_i^* and W_i^* .

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Thanks!