The Schur-Positivity Poset

Peter McNamara

Includes joint work with Stephanie van Willigenburg

Algebra etc. Seminar
Bucknell University
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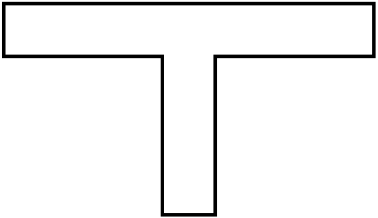
Papers available from
`www.facstaff.bucknell.edu/pm040/`

Outline

Shape of talk:



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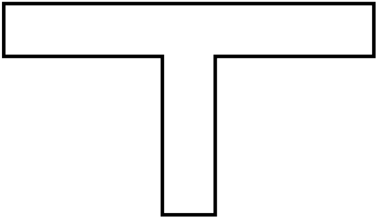


- Where to go from the basics: one direction
- Quick review: skew Schur functions, Schur-positivity
- Project 1: Necessary conditions for Schur-positivity
- Project 2: Special subposets
- Project 3 (Focus): Equality of skew Schur functions
- Open problems

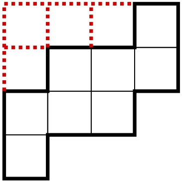
One direction: generalizations/variations

- Quasi-symmetric functions
- Hall-Littlewood polynomials (coeffs in $\mathbb{Q}(t)$)
- Macdonald polynomials (coeffs in $\mathbb{Q}(q, t)$)
- Toric/Cylindric skew Schur functions
- Grothendieck polynomials
- Non-commutative symmetric functions

Each of these areas has a rich theory and important connections to other fields.

Review: Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4, 4, 3, 1)/(3, 1)$



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  Example:
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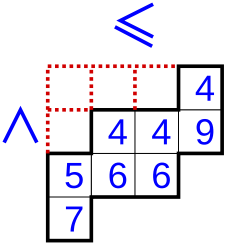
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

Review: Schur-positivity

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

$$s_{\mu} s_{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}.$$

$c_{\mu\nu}^{\lambda}$ is the number of SSYT of shape λ/μ and content ν whose reverse reading word is a ballot sequence.

Key: $c_{\mu\nu}^{\lambda} \geq 0$.

$s_{\lambda/\mu}$ and $s_{\mu}s_{\nu}$ are *Schur-positive*.

Big Question

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

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When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ or $s_{\lambda} s_{\nu} - s_{\sigma} s_{\tau}$ Schur-positive?

Big Question

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

$$s_{\lambda} s_{\mu} = \sum_{\nu} c^{\nu}_{\lambda\mu} s_{\nu}.$$

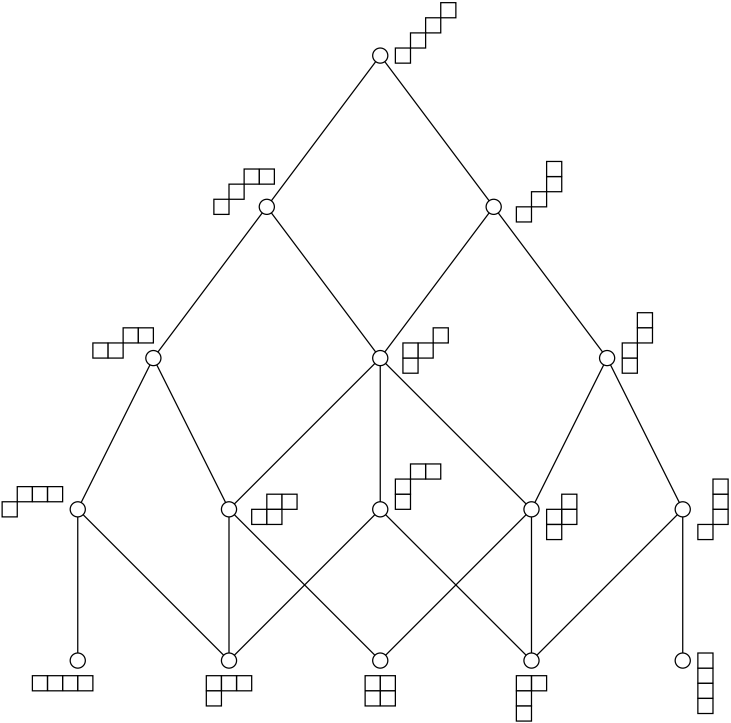
When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ or $s_{\lambda} s_{\nu} - s_{\sigma} s_{\tau}$ Schur-positive?

(Note: latter is a special case of the former.)

Goal: Characterize the shapes $\lambda, \mu, \sigma, \tau$ that make $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive.

Example

$n = 4$ Schur-positivity partially ordered set (Schur-positivity poset) on board.



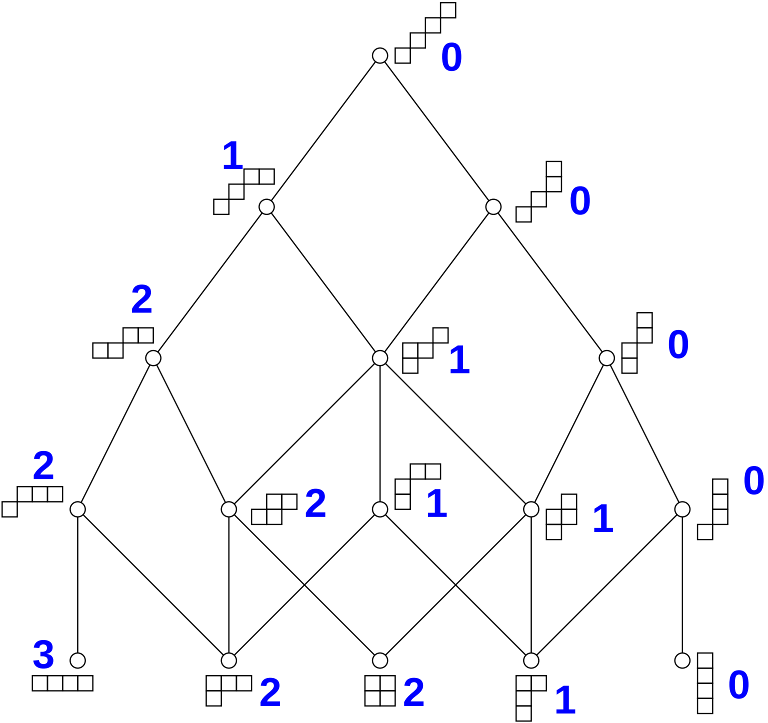
Theorem

If $s_A - s_B \geq 0$ (i.e. Schur-positive), then

$$\text{the number of } m \times n \text{ rectangles fitting inside } A \leq \text{the number of } m \times n \text{ rectangles fitting inside } B.$$

Example

$m = 1, n = 2$ in example on board.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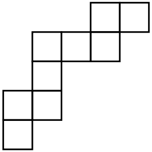
Project 2: Subposet of multiplicity-free ribbons

Joint work with Stephanie van Willigenburg

Definition

A ribbon is a connected skew shape containing no 2×2 rectangle.

Example



Indexed as 23121.

Joint work with Stephanie van Willigenburg

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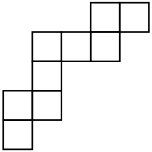

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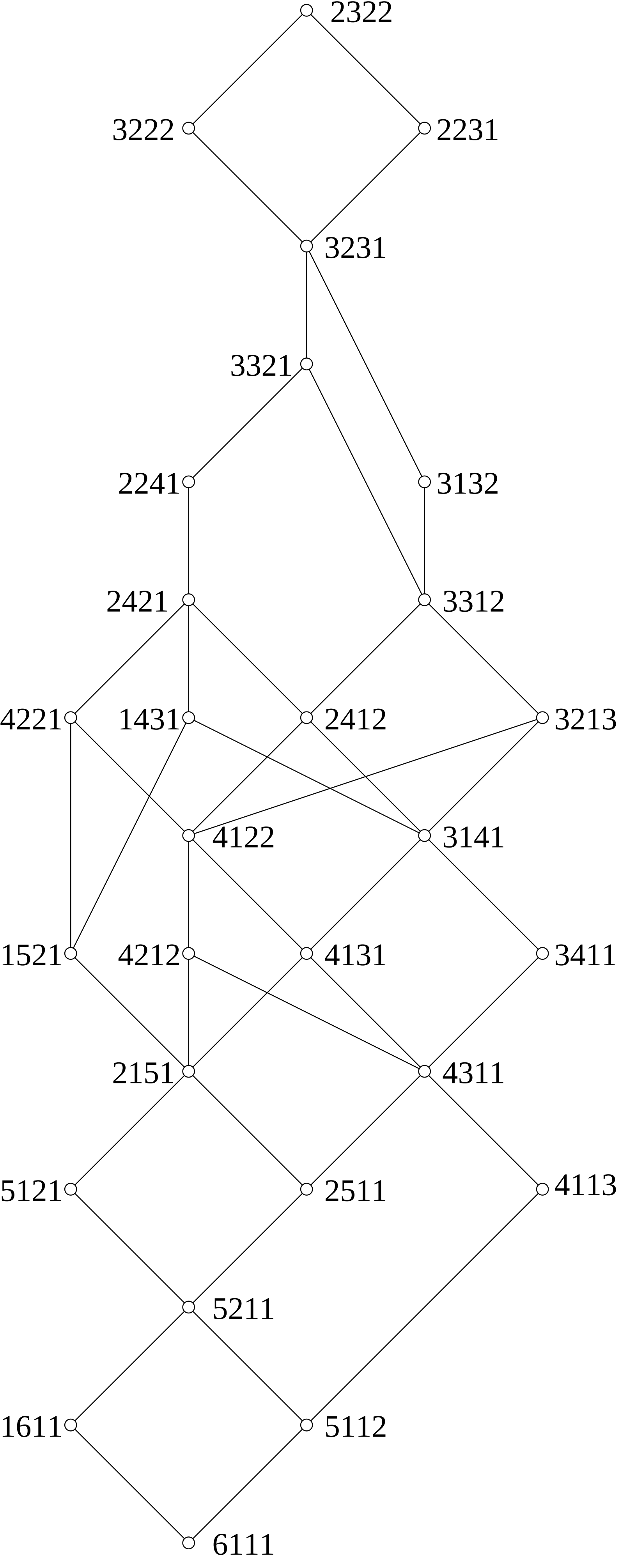
Suffices to fix #boxes and #rows.

Example

Ribbons$(9, 4)$ on board.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4131
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2151
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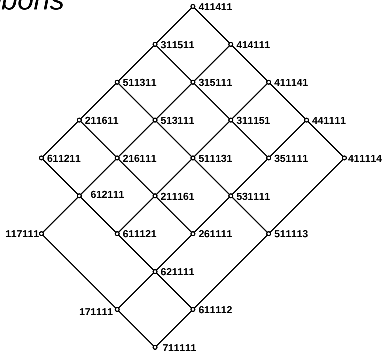
5121
2511
4113
5211
1611
5112
6111

Definition

A skew shape A is said to be multiplicity-free if, when s_A is expanded as a linear combination of Schur functions, each coefficient is 0 or 1.

Theorem

The poset of multiplicity-free ribbons is always of the form



and is a convex subposet of the appropriate Schur-positivity poset.

Project 3: Equality of Schur functions

Joint work with Stephanie van Willigenburg

Example

Look at Schur-positivity poset for $n = 4$ again.
There appear to be some skew shapes missing.

Joint work with Stephanie van Willigenburg

Example

Look at Schur-positivity poset for $n = 4$ again.
There appear to be some skew shapes missing.

Question

When do two skew shapes have the same skew Schur function? Can we classify this in terms of the shapes of the skew shapes?

An answer to this question could be thought of as a classification of skew Schur functions.

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

The HDL series

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- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

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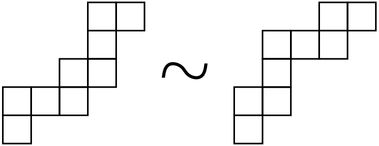
Complete classification of equality of ribbon Schur functions

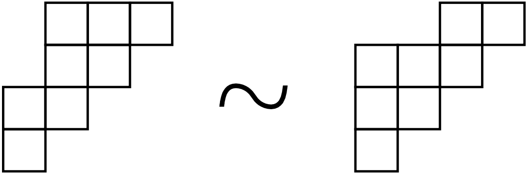


- HDL II: Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006):
  - It's enough to understand the equalities among *connected* skew diagrams.
  - 3 operations for generating skew diagrams with equal skew Schur functions.
  - For $\#\textit{boxes} \leq 18$, there are 6 examples that escape explanation.
  - Necessary conditions, but of a different flavor. (This was the inspiration for Project 1.)

- HDL III: McN., Steph van Willigenburg (2006):
  - An operation that encompasses the operation of HDL I and the three operations of HDL II.
  - Theorem that generalizes all previous results.
    Explains all equivalences where $\#boxes \leq 20$.
  - Conjecture for necessary and sufficient conditions for $s_{\lambda/\alpha} = s_{\mu/\beta}$.
    Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes) D, E.
If $s_D = s_E$, we will write $D \sim E$.

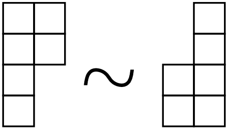
Example



We want to classify all equivalences classes, thereby classifying all skew Schur functions.

The basic building block

Stanley's *Enumerative Combinatorics, Volume II*, Exercise 7.56(a)



Theorem

$D \sim D^*$, *where* D^* *denotes* D *rotated by* 180°.

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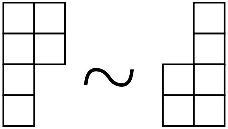

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Goal: Use this equivalence to build other skew equivalences.

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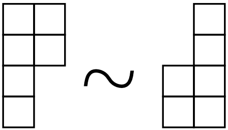


Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

Goal: Use this equivalence to build other skew equivalences.

Where we're headed:

Theorem

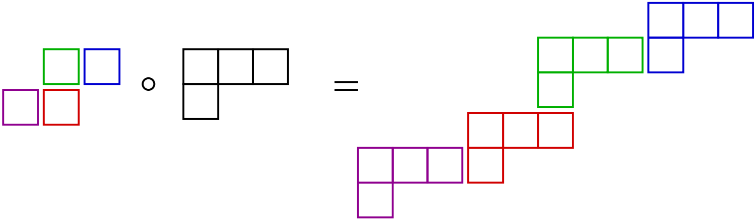
Suppose we have skew diagrams D, D' and E satisfying certain assumptions. If $D \sim D'$ then

$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$

Main definition: composition of skew diagrams.

Composition of skew diagrams

$D \circ E =$



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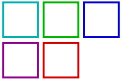
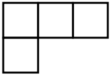
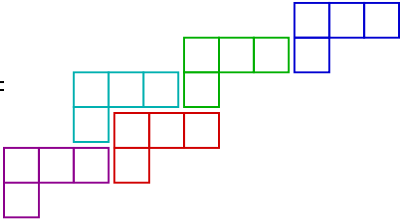
$D \circ E =$  $\circ$  $=$ 

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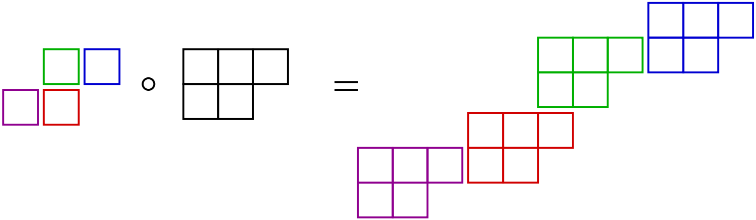
$D \circ E =$ 

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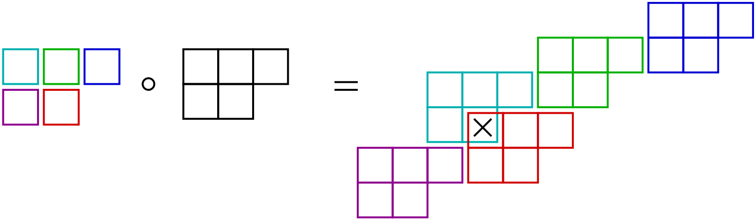

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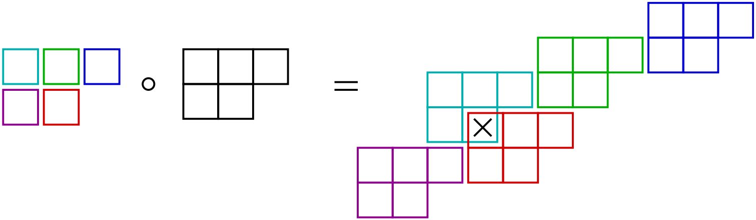
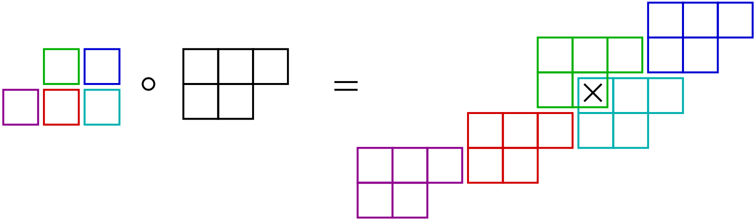
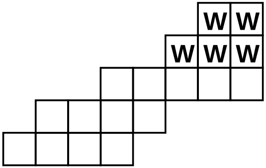
$D \circ E =$ 

Theorem. *If $D \sim D'$, then*

$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$



A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



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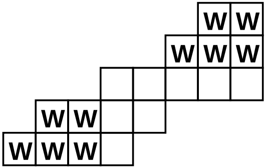
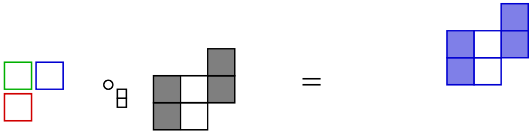
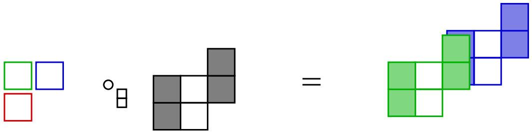


Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

$D \circ_W E =$ 

Amalgamated Compositions

$D \circ_W E =$ 

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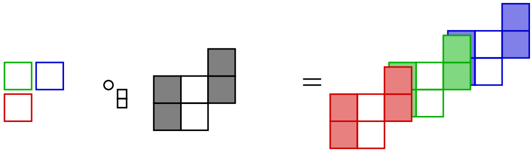

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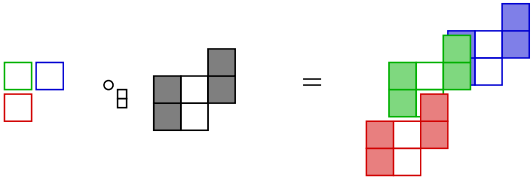

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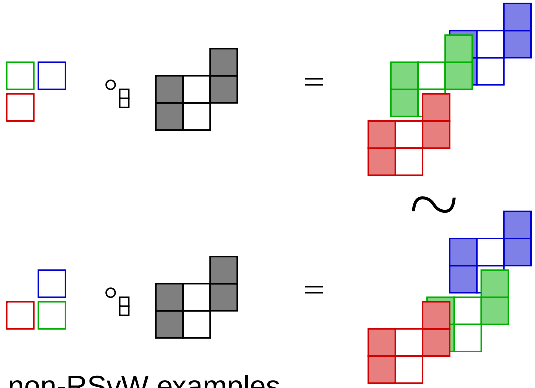

15 boxes: first of the non-RSvW examples

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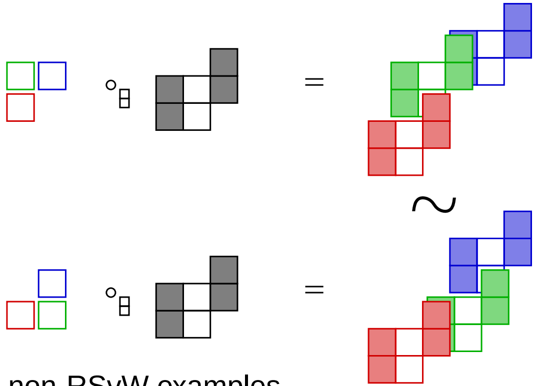
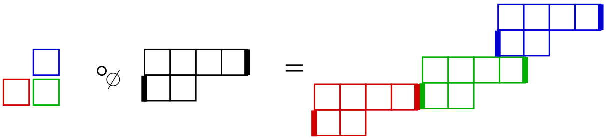


15 boxes: first of the non-RSvW examples

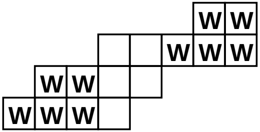
If $W = \emptyset$, we get the regular compositions:

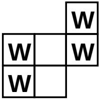


The hypotheses

In $E = WOW$, W and O must satsify certain conditions, all of which are natural expect for:

Unwanted Hypothesis. In $E = WOW$, at least one copy of W has just one cell adjacent to O.





What are the results?

Theorem. Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying all the hypotheses. Then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$

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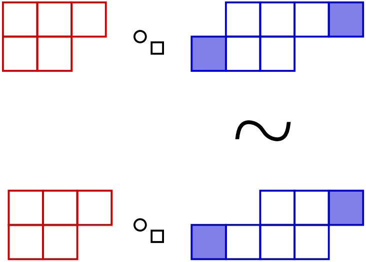

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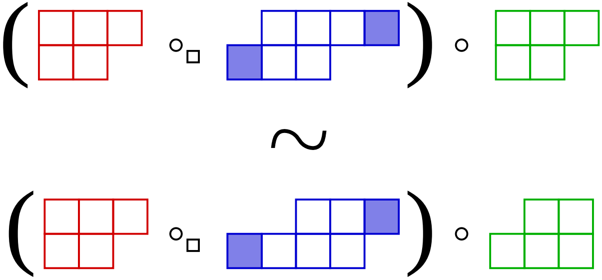


is a skew equivalence with 145 boxes.

A word about the proof

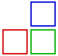
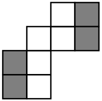
The key: An expression for $s_{D \circ_W E}$ in terms of s_D, s_E, s_W, s_O.

Proof of expression uses:

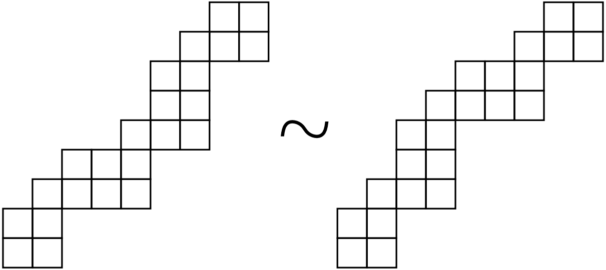
- Hamel-Goulden determinants.
- Sylvester's Determinantal Identity.

Open problems

- Removing Unwanted Hypothesis (at least one copy of W has just one cell adjacent to O).

$D =$  $E =$ 

$D \circ_W E$ has 23 boxes, and $D \circ_W E \sim D^* \circ_W E$:



(Maple-based software of Anders Buch, John Stembridge)

Main open problem

Theorem.
Skew diagrams $E_1, E_2, \ldots, E_r$.
$E_i = W_i O_i W_i$ satisfies all the hypotheses for all i.
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \;\sim\; ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' \,.$$

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Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \text{ where} \end{aligned}$$

- $\circ$ $E_i = W_i O_i W_i$ satisfies the natural hypotheses for all i,
- $\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

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Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \end{aligned}$$ where

- $E_i = W_i O_i W_i$ satisfies the natural hypotheses for all i,
- E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

True when $\#boxes \leq 20$.

Other open problems

- When does $\circ$ preserve Schur-positivity, i.e. if

$$s_A - s_B \geq 0,$$

when is

$$s_{C \circ A} - s_{C \circ B} \geq 0,$$

or

$$s_{A \circ C} - s_{B \circ C} \geq 0 \quad ?$$

- Of the connected elements of the Schur-positivity poset, what's at the top?
- Given a positive linear combination $\sum_\nu a_\nu s_\nu$, how do we tell if it's a skew Schur function?

Summary

- Project 1: Necessary conditions involving rectangles fitting inside shapes.
- Project 2: Subposet of multiplicity-free ribbons is a grid-like chunk in the Schur-positivity poset.
- Project 3: Theorem about skew Schur equivalence that unifies and generalizes all previous results, and allows for a conjecture about the complete answer.