

# The Schur-Positivity Poset

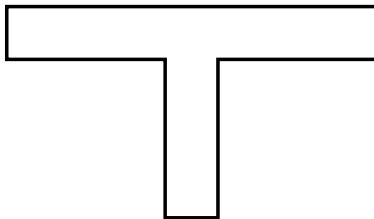
Peter McNamara

Includes joint work with Stephanie van Willigenburg

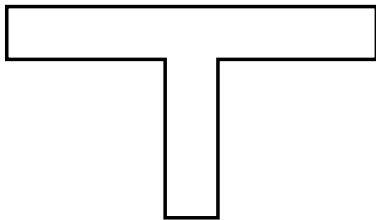
Algebra etc. Seminar  
Bucknell University  
20 September 2007

Papers available from  
[www.facstaff.bucknell.edu/pm040/](http://www.facstaff.bucknell.edu/pm040/)

Shape of talk:



Shape of talk:



- ▶ Where to go from the basics: one direction
- ▶ Quick review: skew Schur functions, Schur-positivity
- ▶ Project 1: Necessary conditions for Schur-positivity
- ▶ Project 2: Special subposets
- ▶ Project 3 (Focus): Equality of skew Schur functions
- ▶ Open problems

# One direction: generalizations/variations

- ▶ Quasi-symmetric functions
- ▶ Hall-Littlewood polynomials (coeffs in  $\mathbb{Q}(t)$ )
- ▶ Macdonald polynomials (coeffs in  $\mathbb{Q}(q, t)$ )
- ▶ Toric/Cylindric skew Schur functions
- ▶ Grothendieck polynomials
- ▶ Non-commutative symmetric functions

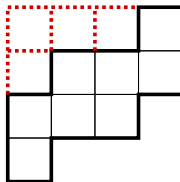
Each of these areas has a rich theory and important connections to other fields.

# Review: Skew Schur functions

- ▶ Partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$
- ▶  $\mu$  fits inside  $\lambda$ .
- ▶ Young diagram.

Example:

$$\lambda/\mu = (4, 4, 3, 1)/(3, 1)$$



# Review: Skew Schur functions

▶ Partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

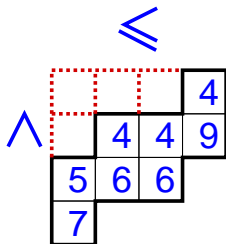
▶  $\mu$  fits inside  $\lambda$ .

▶ Young diagram.

Example:

$$\lambda/\mu = (4, 4, 3, 1)/(3, 1)$$

▶ Semistandard Young tableau (SSYT)



The **skew** Schur function  $s_{\lambda/\mu}$  in the variables  $x = (x_1, x_2, \dots)$  is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \dots$$

**Example**

$$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \dots$$

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

$$s_{\mu} s_{\nu} = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}.$$

$c_{\mu\nu}^{\lambda}$  is the number of SSYT of shape  $\lambda/\mu$  and content  $\nu$  whose reverse reading word is a ballot sequence.

**Key:**  $c_{\mu\nu}^{\lambda} \geq 0$ .

$s_{\lambda/\mu}$  and  $s_{\mu} s_{\nu}$  are *Schur-positive*.

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

$$s_{\lambda} s_{\mu} = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}.$$



# Big Question

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

$$s_{\lambda} s_{\mu} = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}.$$

When is  $s_{\lambda/\mu} - s_{\sigma/\tau}$  or  $s_{\lambda} s_{\nu} - s_{\sigma} s_{\tau}$  Schur-positive?

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

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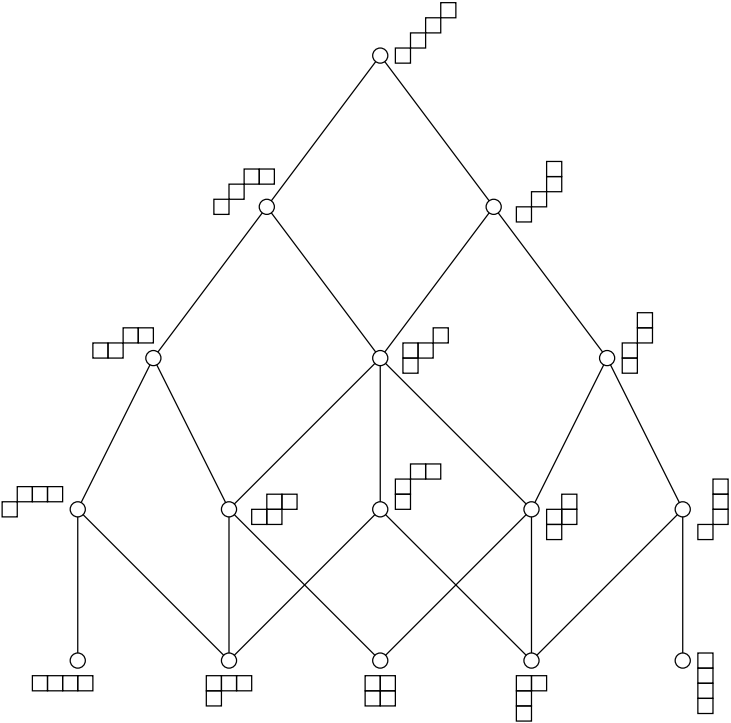
When is  $s_{\lambda/\mu} - s_{\sigma/\tau}$  or  $s_{\lambda} s_{\nu} - s_{\sigma} s_{\tau}$  Schur-positive?

(Note: latter is a special case of the former.)

**Goal:** Characterize the shapes  $\lambda, \mu, \sigma, \tau$  that make  $s_{\lambda/\mu} - s_{\sigma/\tau}$  Schur-positive.

## Example

$n = 4$  Schur-positivity partially ordered set (Schur-positivity poset) on board.



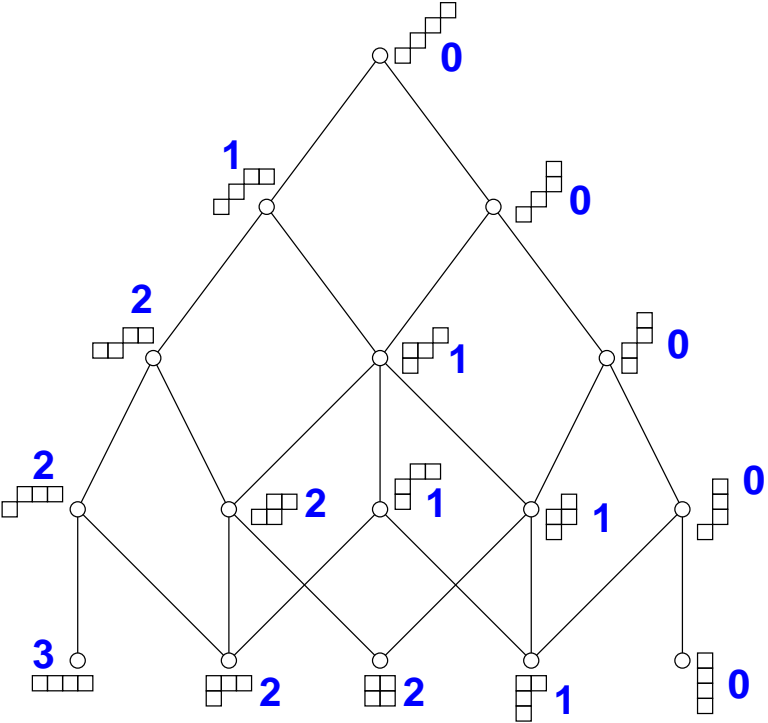
## Theorem

If  $s_A - s_B \geq 0$  (i.e. Schur-positive), then

*the number of  $m \times n$  rectangles fitting inside  $A \leq$   
the number of  $m \times n$  rectangles fitting inside  $B$ .*

## Example

$m = 1, n = 2$  in example on board.



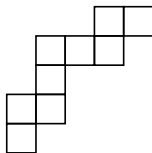
# Project 2: Subposet of multiplicity-free ribbons

Joint work with Stephanie van Willigenburg

## Definition

A **ribbon** is a connected skew shape containing no  $2 \times 2$  rectangle.

## Example



Indexed as 23121.

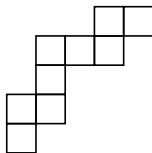
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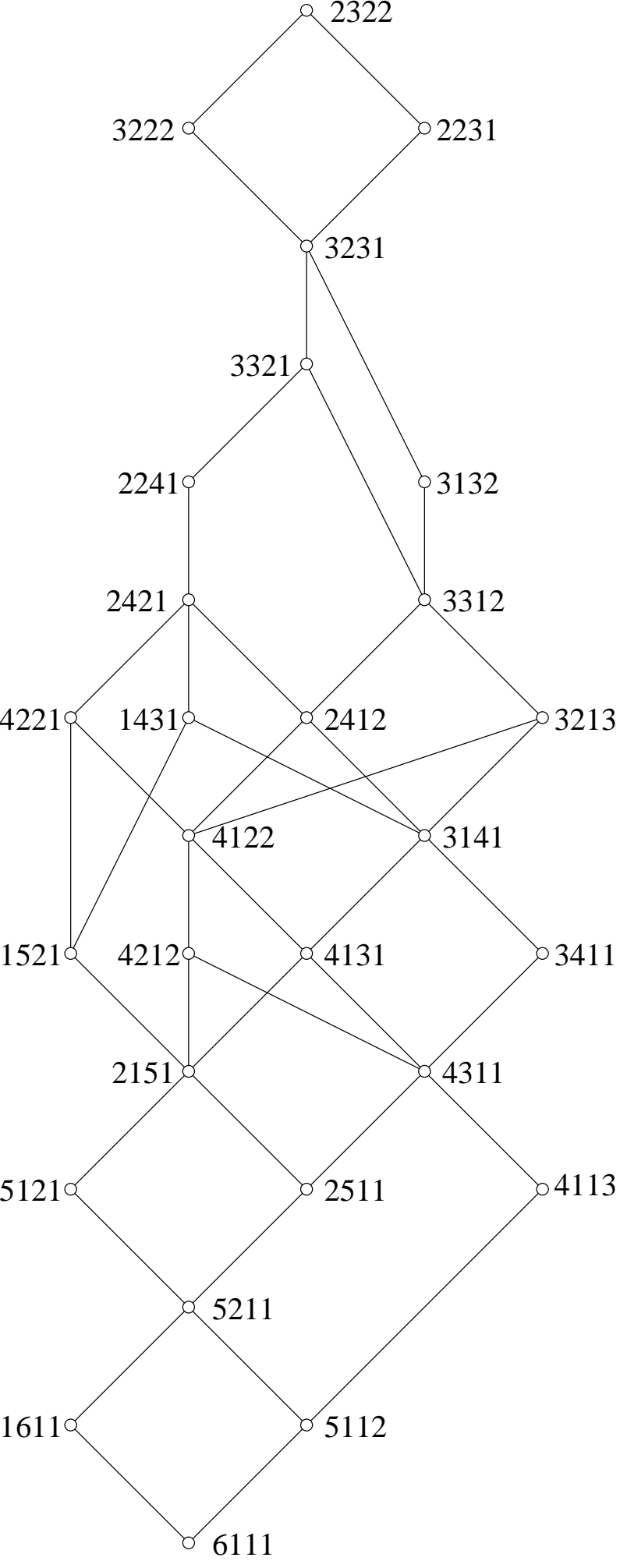


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Suffices to fix #boxes and #rows.

## Example

Ribbons(9, 4) on board.





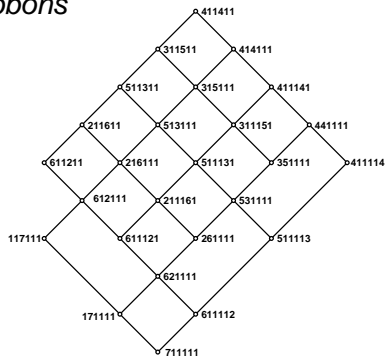
# Project 2: Subset of multiplicity-free ribbons

## Definition

A skew shape  $A$  is said to be **multiplicity-free** if, when  $s_A$  is expanded as a linear combination of Schur functions, each coefficient is 0 or 1.

## Theorem

*The poset of multiplicity-free ribbons is always of the form*



*and is a convex subposet of the appropriate Schur-positivity poset.*

# Project 3: Equality of Schur functions

Joint work with Stephanie van Willigenburg

## Example

Look at Schur-positivity poset for  $n = 4$  again.  
There appear to be some skew shapes missing.

Joint work with Stephanie van Willigenburg

## Example

Look at Schur-positivity poset for  $n = 4$  again.  
There appear to be some skew shapes missing.

## Question

*When do two skew shapes have the same skew Schur function? Can we classify this in terms of the shapes of the skew shapes?*

An answer to this question could be thought of as a classification of skew Schur functions.

**Big Question:** When is  $s_{\lambda/\alpha} = s_{\mu/\beta}$  ?

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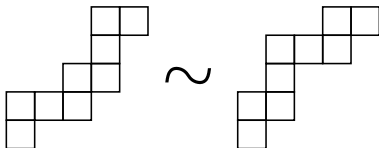


**Big Question:** When is  $s_{\lambda/\alpha} = s_{\mu/\beta}$  ?

- ▶ Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):



Complete classification of equality of **ribbon** Schur functions



- ▶ HDL II: Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006):
  - ▶ It's enough to understand the equalities among *connected* skew diagrams.
  - ▶ 3 operations for generating skew diagrams with equal skew Schur functions.
  - ▶ For  $\#boxes \leq 18$ , there are 6 examples that escape explanation.
  - ▶ Necessary conditions, but of a different flavor. (This was the inspiration for Project 1.)

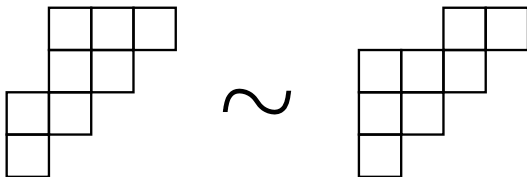


- ▶ HDL III: McN., Steph van Willigenburg (2006):
  - ▶ An operation that encompasses the operation of HDL I and the three operations of HDL II.
  - ▶ Theorem that generalizes all previous results.  
Explains all equivalences where  $\#boxes \leq 20$ .
  - ▶ Conjecture for necessary and sufficient conditions for  $s_{\lambda/\alpha} = s_{\mu/\beta}$ .  
Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes)  $D, E$ .

If  $s_D = s_E$ , we will write  $D \sim E$ .

## Example



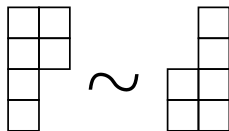
We want to classify all equivalence classes, thereby classifying all skew Schur functions.

# The basic building block

Stanley's *Enumerative Combinatorics, Volume II*,  
Exercise 7.56(a)

## Theorem

$D \sim D^*$ , where  $D^*$  denotes  $D$  rotated by  $180^\circ$ .



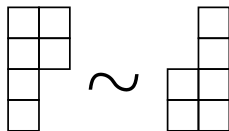
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**Goal:** Use this equivalence to build other skew equivalences.

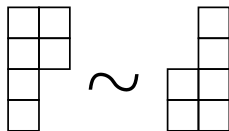


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**Goal:** Use this equivalence to build other skew equivalences.

Where we're headed:

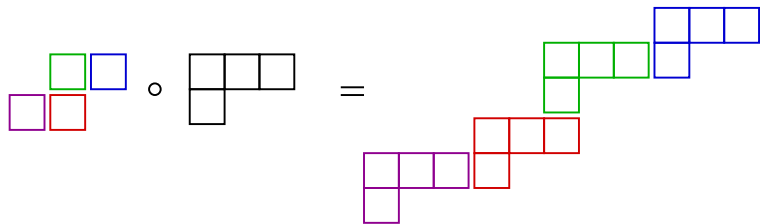
## Theorem

Suppose we have skew diagrams  $D$ ,  $D'$  and  $E$  satisfying certain assumptions. If  $D \sim D'$  then

$$D' \circ E \sim D \circ E \sim D \circ E^*.$$

Main definition: composition of skew diagrams.

# Composition of skew diagrams

$$D \circ E =$$


The diagram illustrates the composition of two skew diagrams,  $D$  and  $E$ , resulting in a larger skew diagram. The composition is shown as  $D \circ E =$  followed by the diagrams.

Diagram  $D$  (left) consists of a 2x2 grid of boxes. The top row has a purple box, a red box, a green box, and a blue box. The bottom row has a purple box and a red box.

Diagram  $E$  (middle) consists of a 2x3 grid of black boxes. The top row has three boxes, and the bottom row has one box on the left.

The result (right) is a larger skew diagram composed of the boxes from  $D$  and  $E$  arranged in a 4x6 grid. The top row has a purple box, a purple box, a purple box, a red box, a red box, and a red box. The second row has a purple box, a red box, a red box, a red box, a green box, and a green box. The third row has a green box, a blue box, a blue box, and a blue box. The fourth row has a blue box.

# Composition of skew diagrams

$$D \circ E = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$$

The diagram on the left shows the composition of two skew diagrams. The first diagram,  $D$ , is a skew diagram with three rows: the top row has three boxes (cyan, green, blue), the second row has two boxes (purple, red), and the third row is empty. The second diagram,  $E$ , is a skew diagram with two rows: the top row has three boxes (black), and the second row has one box (black). The result,  $D \circ E$ , is a skew diagram with four rows: the top row has four boxes (cyan, green, blue, black), the second row has three boxes (cyan, red, black), the third row has three boxes (purple, red, black), and the fourth row has one box (purple).

# Composition of skew diagrams

$$D \circ E =$$

The diagram illustrates the composition of two skew diagrams,  $D$  and  $E$ , resulting in a new skew diagram. The left side shows  $D$  (a purple square) and  $E$  (a red square) with a green square above the red one and a blue square to the right of the green one. This is followed by a black skew diagram consisting of two rows of three squares each, with the bottom row shifted one square to the right. An equals sign follows. On the right, the resulting diagram is a staircase shape where the purple, red, green, and blue components are shifted and combined into a single structure.



# Composition of skew diagrams

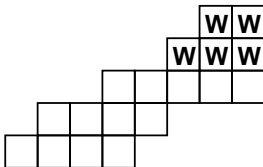
$$D \circ E = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \end{array}$$

The diagram on the right shows the composition of the two diagrams. The top row consists of four squares: purple, cyan, green, and blue. The second row consists of four squares: purple, cyan, red, and blue. The third row consists of two squares: purple and red. The fourth row consists of two squares: purple and red. The fifth row consists of two squares: cyan and red. The sixth row consists of two squares: cyan and red. The seventh row consists of two squares: green and blue. The eighth row consists of two squares: green and blue. The ninth row consists of two squares: blue and blue. The tenth row consists of two squares: blue and blue. The eleventh row consists of two squares: blue and blue. The twelfth row consists of two squares: blue and blue. The thirteenth row consists of two squares: blue and blue. The fourteenth row consists of two squares: blue and blue. The fifteenth row consists of two squares: blue and blue. The sixteenth row consists of two squares: blue and blue. The seventeenth row consists of two squares: blue and blue. The eighteenth row consists of two squares: blue and blue. The nineteenth row consists of two squares: blue and blue. The twentieth row consists of two squares: blue and blue. The twenty-first row consists of two squares: blue and blue. The twenty-second row consists of two squares: blue and blue. The twenty-third row consists of two squares: blue and blue. The twenty-fourth row consists of two squares: blue and blue. The twenty-fifth row consists of two squares: blue and blue. The twenty-sixth row consists of two squares: blue and blue. The twenty-seventh row consists of two squares: blue and blue. The twenty-eighth row consists of two squares: blue and blue. The twenty-ninth row consists of two squares: blue and blue. The thirtieth row consists of two squares: blue and blue. The thirtieth row contains a black 'X' in the cyan square.



# Amalgamated Compositions

A skew diagram  $W$  *lies in the top* of a skew diagram  $E$  if  $W$  appears as a connected subdiagram of  $E$  that includes the northeasternmost cell of  $E$ .





# Amalgamated Compositions

$$D \circ_W E =$$

The diagram shows the composition  $D \circ_W E$ . On the left,  $D$  is represented by three boxes: a green box, a blue box, and a red box. On the right,  $E$  is represented by six boxes: a gray box, a white box, a gray box, a gray box, a white box, and a gray box. The composition is indicated by a small circle and a vertical bar. The result is a composition of six blue boxes.

# Amalgamated Compositions

$$D \circ_W E =$$

The diagram illustrates the amalgamated composition  $D \circ_W E$ . On the left, composition  $D$  consists of three boxes: a green box, a blue box, and a red box. Composition  $E$  consists of six boxes: a gray box, a white box, a gray box, a gray box, a white box, and a gray box. The result is a composition with nine boxes: the first three are green, the next three are white, and the last three are blue.

# Amalgamated Compositions

$$D \circ_W E =$$

The diagram illustrates the amalgamated composition  $D \circ_W E$ . On the left, composition  $D$  consists of two boxes: a green box on top and a red box on the bottom. Composition  $E$  consists of a 2x2 grid of white boxes, with a gray box on top-right and another gray box on top-right of that. An arrow points to the result, which is a composition where the boxes are interleaved: a 2x2 grid of red boxes, followed by a 2x2 grid of green boxes, followed by a 2x2 grid of blue boxes.

# Amalgamated Compositions

$$D \circ_W E =$$

The diagram illustrates the amalgamated composition  $D \circ_W E$ . On the left, composition  $D$  consists of a green square above a red square, and a blue square to the right of the green square. Composition  $E$  consists of a 2x2 grid of gray and white squares, with a gray square above the top-right white square. The composition symbol  $\circ$  is shown with a small white square below it. On the right, the result is a composition where the green and red squares from  $D$  are placed on top of the white squares from  $E$ , and the blue square from  $D$  is placed to the right of the top-right white square from  $E$ .



# Amalgamated Compositions

$$\begin{array}{c}
 D \circ_W E = \\
 \begin{array}{ccc}
 \begin{array}{|c|c|} \hline \color{green}{\square} & \color{blue}{\square} \\ \hline \color{red}{\square} & \\ \hline \end{array} & \circ_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} & \begin{array}{|c|c|c|} \hline \color{gray}{\square} & \square & \color{gray}{\square} \\ \hline \color{gray}{\square} & \square & \\ \hline \end{array} & = & \begin{array}{|c|c|c|c|} \hline & \color{green}{\square} & & \color{blue}{\square} \\ \hline \color{green}{\square} & \square & \color{green}{\square} & \color{blue}{\square} \\ \hline \color{red}{\square} & \square & \color{red}{\square} & \\ \hline \color{red}{\square} & \square & & \\ \hline \end{array} \\
 \\
 \begin{array}{ccc}
 \begin{array}{|c|c|} \hline \color{red}{\square} & \color{blue}{\square} \\ \hline \color{red}{\square} & \color{green}{\square} \\ \hline \end{array} & \circ_{\begin{array}{|c|} \hline \square \\ \hline \end{array}} & \begin{array}{|c|c|c|} \hline \color{gray}{\square} & \square & \color{gray}{\square} \\ \hline \color{gray}{\square} & \square & \\ \hline \end{array} & = & \begin{array}{|c|c|c|c|} \hline & \color{blue}{\square} & & \color{blue}{\square} \\ \hline \color{blue}{\square} & \square & \color{blue}{\square} & \\ \hline \color{red}{\square} & \square & \color{red}{\square} & \color{green}{\square} \\ \hline \color{red}{\square} & \square & & \color{green}{\square} \\ \hline \end{array} \\
 \end{array} \sim \begin{array}{|c|c|c|c|} \hline & \color{blue}{\square} & & \color{blue}{\square} \\ \hline \color{blue}{\square} & \square & \color{blue}{\square} & \\ \hline \color{red}{\square} & \square & \color{red}{\square} & \color{green}{\square} \\ \hline \color{red}{\square} & \square & & \color{green}{\square} \\ \hline \end{array}
 \end{array}$$

15 boxes: first of the non-RSvW examples

# Amalgamated Compositions

$$D \circ_W E = \begin{array}{c} \square \square \\ \square \end{array} \circ_{\begin{array}{c} \square \\ \square \end{array}} \begin{array}{c} \square \\ \square \square \end{array} = \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \square \square \end{array} \sim \begin{array}{c} \square \square \square \\ \square \square \square \\ \square \square \square \end{array}$$

The diagram shows the amalgamation of composition  $D$  (two boxes, top blue, bottom red) and composition  $E$  (two boxes, top grey, bottom white) using the operation  $\circ_W$  with  $W = \{\square\}$ . The result is a composition of 15 boxes, which is shown to be equivalent to another composition of 15 boxes where the boxes are interleaved.

15 boxes: first of the non-RSvW examples

If  $W = \emptyset$ , we get the regular compositions:

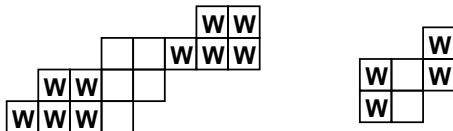
$$\begin{array}{c} \square \\ \square \square \end{array} \circ_{\emptyset} \begin{array}{c} \square \square \square \square \\ \square \square \square \end{array} = \begin{array}{c} \square \square \square \square \\ \square \square \square \square \\ \square \square \square \square \end{array}$$

The diagram shows the amalgamation of composition  $D$  (two boxes, top blue, bottom red) and composition  $E$  (two rows of four boxes, top blue, bottom white) using the operation  $\circ_{\emptyset}$ . The result is a composition of 15 boxes arranged in three rows of five boxes each, with the top row blue, the middle row green, and the bottom row red.

# The hypotheses

In  $E = WOW$ ,  $W$  and  $O$  must satisfy certain conditions, all of which are natural expect for:

**Unwanted Hypothesis.** In  $E = WOW$ , at least one copy of  $W$  has just one cell adjacent to  $O$ .



# What are the results?

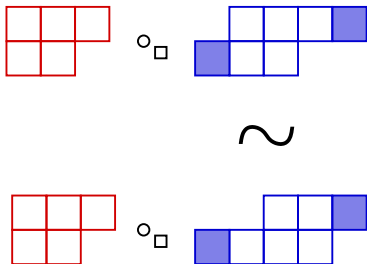
**Theorem.** Suppose we have skew diagrams  $D, D'$  with  $D \sim D'$  and  $E = \text{WOW}$  satisfying all the hypotheses. Then

$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$

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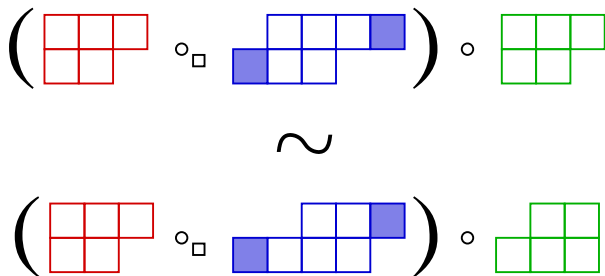
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$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$



is a skew equivalence with 145 boxes.

The key: An expression for  $s_{D \circ_W E}$  in terms of  $s_D, s_E, s_W, s_O$ .

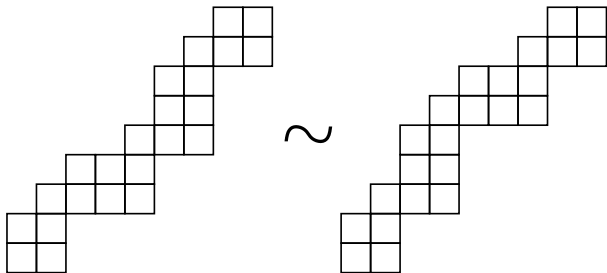
Proof of expression uses:

- ▶ Hamel-Goulden determinants.
- ▶ Sylvester's Determinantal Identity.

- ▶ Removing Unwanted Hypothesis (at least one copy of  $W$  has just one cell adjacent to  $O$ ).

$$D = \begin{array}{c} \square \\ \square \end{array} \quad E = \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array}$$

$D \circ_W E$  has 23 boxes, and  $D \circ_W E \sim D^* \circ_W E$ :



(Maple-based software of Anders Buch, John Stembridge)



# Main open problem

## Theorem.

Skew diagrams  $E_1, E_2, \dots, E_r$ .

$E_i = W_i O_i W_i$  satisfies all the hypotheses for all  $i$ .

$E'_i$  and  $W'_i$  denote either  $E_i$  and  $W_i$ , or  $E_i^*$  and  $W_i^*$ .

Then

$$((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r.$$

# Main open problem

## Theorem.

Skew diagrams  $E_1, E_2, \dots, E_r$ .

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**Conjecture.** [McN, van Willigenburg; inspired by main result of BTvW]

Two skew diagrams  $E$  and  $E'$  satisfy  $E \sim E'$  if and only if, for some  $r$ ,

$$\begin{aligned} E &= ((\cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3) \cdots) \circ_{W_r} E'_r, \text{ where} \end{aligned}$$

- $E_i = W_i O_i W_i$  satisfies the natural hypotheses for all  $i$ ,
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True when #boxes  $\leq 20$ .

- ▶ When does  $\circ$  preserve Schur-positivity, i.e. if

$$s_A - s_B \geq 0,$$

when is

$$s_{C \circ A} - s_{C \circ B} \geq 0,$$

or

$$s_{A \circ C} - s_{B \circ C} \geq 0 ?$$

- ▶ Of the connected elements of the Schur-positivity poset, what's at the top?
- ▶ Given a positive linear combination  $\sum_{\nu} a_{\nu} s_{\nu}$ , how do we tell if it's a skew Schur function?

- ▶ Project 1: Necessary conditions involving rectangles fitting inside shapes.
- ▶ Project 2: Subset of multiplicity-free ribbons is a grid-like chunk in the Schur-positivity poset.
- ▶ Project 3: Theorem about skew Schur equivalence that unifies and generalizes all previous results, and allows for a conjecture about the complete answer.