Computer Methods for Siegel Modular Forms

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Ingredients

- Arithmetic group
- Upper half-plane
- Automorphy factor
- Growth condition

Ingredients

- Arithmetic group: finite index subgroup

$$\Gamma^n \subset Sp_{2n}(\mathbb{Z}) = \{M \in GL_{2n}(\mathbb{Z}) : M^t J M = J\}, J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

- Upper half-plane: $\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}) : Z^t = Z, \mathrm{Im}(Z) > 0\}$.
- Let $\rho : GL(n, \mathbb{C}) \to GL(V)$ be a rational representation of a finite dimensional vector space.

Siegel Modular Forms

Let $M_\rho(\Gamma^n) = M^n_\rho$ be the space of **Siegel modular forms of weight ρ and degree n**. I.e., $F \in M^n_\rho$ iff

- $F : \mathfrak{h}_n \to V$ is holomorphic,
- $F\left((AZ+B)(CZ+D)^{-1}\right) = \rho(CZ+D)F(Z)\ \forall \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^n$
- For $n = 1$, we need to assert a growth condition.

Siegel Modular Forms – Fourier Expansions

- Let $Q := \{f = [a, b, c] = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y] \ : \ f \geq 0\}$.
- We have

$$\begin{aligned} F(Z) &= \sum_{f=[a,b,c]\in Q} C(f) e\left(\operatorname{tr}(ZM_f)\right) \\ &= \sum_{f=[a,b,c]\in Q} C(f) e\left(\operatorname{tr}\left(Z\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right)\right) \\ &= \sum_{f=[a,b,c]\in Q} C(f) e\left(a\tau + bz + c\tau'\right) \end{aligned}$$

where $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ ($\tau, \tau' \in \mathbb{H}_1$ and $z \in \mathbb{C}$) and $e(x) = e^{2\pi i x}$.

- If the expansion of F is only supported on positive definite forms, it's a **cusp form**, i.e. $F \in S_\rho^n$.

Classical vs Siegel recap

- When $n = 1$, they are the same.
- Siegel modular forms are multivariate modular forms.
- For degree > 1, the growth condition comes for free (Koecher's principle).
- Fourier expansion supported on matrices.
- For degree > 1 knowing coefficients allows computation of Hecke eigenvalues but **not** vice versa.
- So really, unlike in the classical case, there are two **different** questions:
  1. How do you compute Fourier coefficients?
  2. How do you compute Hecke eigenvalues?

Coefficients: Scalar weight, Level 1, Degree 2

- In this case $\rho = \det(\text{std})^k$ where $\text{std} : GL(2, \mathbb{C}) \to GL(\mathbb{C}^2)$ is inclusion and k is some integer (since ρ is rational).
- Igusa: there are four generators for the ring of Siegel modular forms: an Eisenstein seriesE_4 of weight 4, E_6 of weight 6, and two cusp forms, χ_{10}, χ_{12}, one of weight 10 and one of weight 12.
- Skoruppa: the four generators of even weight can be written down explicitly as Saito-Kurokawa lifts of e_4, e_6, Δ.

Coefficients: Scalar weight, Level 1, Degree 2

- Igusa: a fifth generator χ_{35}.
- Ibukiyama: a formula for this generator

$$\chi_{35} = \frac{1}{(2\pi i)^3} \begin{vmatrix} 4E_4 & 6E_6 & 10\chi_{10} & 12\chi_{12} \\ \frac{\partial E_4}{\partial \tau} & \frac{\partial E_6}{\partial \tau} & \frac{\partial \chi_{10}}{\partial \tau} & \frac{\partial \chi_{12}}{\partial \tau} \\ \frac{\partial E_4}{\partial z} & \frac{\partial E_6}{\partial z} & \frac{\partial \chi_{10}}{\partial z} & \frac{\partial \chi_{12}}{\partial} \\ \frac{\partial E_4}{\partial \tau'} & \frac{\partial E_6}{\partial \tau'} & \frac{\partial \chi_{10}}{\partial \tau'} & \frac{\partial \chi_{12}}{\partial \tau'} \end{vmatrix}.$$

Sage Demo

Coefficients: Scalar weight, Level 1, Degree > 2

- Tsuyumine: A list of 34 generators for the ring of Siegel modular forms of level 1, degree 3 and scalar weight is known. It is not known if they are algebraically independent.
- Poor-Yuen: Computations done in level 1, degree 4 and weights less than 16 via theta series. Everything so far appears to be a lift of some kind.

Several types of congruence subgroups. We restrict our attention to

$$\Gamma_0^{(2)}(N) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) : C \equiv 0 \pmod{N} \right\}.$$

Coefficients: Scalar weight, Level >1, Degree 2

- q be a positive definite quadratic form on a lattice L of rank n.
- Assume L is integral, i.e. $q(L) \subseteq \mathbb{Z}$.
- The theta series of degree 2 associated to L can be defined as

$$\theta_L^{(2)}(Z) := \sum_{(v,w)\in L\times L} e\left(\mathrm{tr}(ZM_{[v,w]})\right)$$

  where $M_{[v,w]} := M_f$ for the binary quadratic form $f = q(xv + yw)$.
- $\theta_L^{(2)}(Z)$ is a Siegel modular form of degree 2 and weight $n/2$, for a certain level and character which can be easily determined.

Coefficients: Scalar weight, Level >1, Degree 2

- Gathering coefficients, write

$$\theta_L^{(2)}(Z) = \sum_f C(f) e(\mathrm{tr}(ZM_f)).$$

- With this notation,

$$C(f) = \# \{(v, w) \in L \times L \ : \ f = q(xv + yw)\}.$$

Sage Demo

- For some particular spaces, one can find lists of generators:

$$\Gamma_0^{(2)}(2),\ \Gamma_0^{(2)}(3,\psi_3) \subset \Gamma_0^{(2)}(3),\ \Gamma_0^{(2)}(4,\psi_4) \subset \Gamma_0^{(2)}(4)$$

- These lists consist of products of theta constants (these give the even weight generators) and forms determined via formulas like the one for χ_{35}.
- For $a, b \in \{0,1\}^2$, set

$$\theta_{a,b}(Z) = \sum_{t\in\mathbb{Z}^2} e\left(\frac{1}{8}Z[2t+a] + \frac{1}{4}{}^t(2t+a)b\right)$$

where $A[B] = {}^tBAB$.

Sage Demo

Our Sage Package

- Co-written by Raum, Ryan, Skoruppa, Tornaría
- Original Cython and Sage Code
- Very general: can handle all the above as well as vector weight Siegel modular forms

Our Sage Package – The Framework

Recall

$$Q := \{f = [a, b, c] = aX^2 + bXY + cY^2 \in \mathbb{Z}[X, Y] \ : \ f \geq 0\}.$$

Note that $f = [a, b, c]$ is positive semidefinite if and only if $b^2 - 4ac \leq 0$ and $a, c \geq 0$. We observe that Q is a semigroup with respect to usual addition. Moreover, GL has a natural left action on Q:

$$\mathrm{GL} \times Q \to Q, \quad (A, f) \mapsto A.f := f\big((X, Y)A\big).$$

We make the following definition:

Definition (Formal Siegel Modular Form)

Let R be a module (or ring) with a GL left action. Then a Siegel modular form C is a map $C : Q \to R$ for which $C(A.f) = A.C(f)$ for all A in a finite index subgroup of GL.

Our Sage Package – Examples

In addition to everything already discussed,

- Saito-Kurokawa lifts: given a Jacobi form of weight k and index 1
$$\phi = \sum_{\substack{D,r\in\mathbb{Z},D\leq 0 \\ D\equiv r^2 \pmod 4}} A_\phi(D) q^{(r^2-D)/4}\zeta^r$$
we define
$$C(f) = C([n,r,m]) := \begin{cases} -\frac{B_{2k}}{4k} A_\phi(0) & f = 0 \\ \sum_{a|(n,r,m)} a^{k-1} A_\phi\left(\frac{r^2-4mn}{a^2}\right) & f = [n,r,m]. \end{cases}$$
- Eisenstein as the S-K lift of the Jacobi Eisenstein series
- Jacobi ϑ-function

Group G	**Monoid** M **with** G**-action**	**Module** R **with** G**-action**	**Type of automorphic form**
1	$\mathbb{Z}_{\geq 0}$	$\mathbb{F}$	elliptic modular forms
GL	Q	$\mathbb{F}[X, Y]_j(\chi)$	vector-valued Siegel modular forms of degree 2
$\mathrm{GL}(n, \mathbb{Z})$	set of semi-positive definite integral quadratic forms f in n variables, $(g, f) \mapsto f\big((X_1, \ldots, X_n)g\big)$	$\mathbb{F}[X_1, \ldots, X_n]_j$, $(g, p) \mapsto \det(g)^k\, p(\big((X_1, \ldots, X_n)g\big)$	vector-valued Siegel modular forms of degree n and weight k
$\mathbb{Z}_L^*$	set of totally positive or zero elements in the inverse different of L, $(g, a) \mapsto g^2 a$	$\mathbb{F}$, $(g, r) \mapsto N(g)^k r$	Hilbert modular forms of (parallel) weight k over a totally real number field L
$GL(n, \mathbb{Z}_L)$	set of semi-positive definite integal hermitian forms f over L with n variables, $(g, f) \mapsto f\big((X_1, \ldots, X_n)g\big)$	$\mathbb{F}$, $(g, p) \mapsto \det(g)^k p$	Hermitian modular forms over the imaginary quadratic field L
$\mathbb{Z}_L^* \ltimes 2m\mathbb{Z}_L$	set of (D, r) in $\vartheta^{-2} \times \vartheta^{-1}$ such that $D \equiv r^2 \bmod 4m\vartheta^{-1}$, $D = 0$ or $-D \gg 0$ (where ϑ denotes the different of L), $\big((\varepsilon, x), (D, r)\big) \mapsto \big((\varepsilon^2 D, \varepsilon(r + x))\big)$	$\mathbb{F}$, $\big((\varepsilon, x), r\big) \mapsto N(\varepsilon)^k r$	Jacobi forms of weight k and index $m \gg 0$ (in ϑ^{-1}) over a number field L

Hecke eigenvalues via Coefficients

If F is a Hecke eigenform, then

$$\lambda_F(p)C([1,1,1]) = C([p,p,p]) + p^{k-2}\left(1 + \left(\frac{p}{3}\right)\right)$$

and for
$\Lambda_p := \lambda_F(p)^2 - \lambda_F(p)p^{k-2}\left(1+\left(\frac{p}{3}\right)\right) - p^{2k-3} + p^{2k-4}\left(\left(\frac{p}{3}\right)+\left(\frac{p}{3}\right)\right)^2$
we have

$$\lambda_F(p^2)C([1,1,1]) = \Lambda_p C([1,1,1]) - p^{k-2}C([1,p,p]^2) - p^{k-2} \sum_{\substack{\nu \bmod p \\ 1+\nu+\nu^2 \not\equiv 0 \bmod p}} C([1+\nu+\nu^2, p(1+2\nu), p^2])$$

- Computing the eigenvalue $\lambda_F(p)$ requires the computation of coefficients up to discriminant $3p^2$.
- Computing the eigenvalue $\lambda_F(p^2)$ requires the computation of coefficients up to discriminant $O(p^4)$.

Maeda-type Conjecture

For all even weights except 24 and 26, the characteristic polynomial of the Hecke operator T_2 is irreducible acting on the space of Siegel modular forms that are not Saito-Kurokawa lifts. This has been checked using this package for weights up to 88.

Sage Demo

Hecke eigenvalues, other methods

- Cunningham-Dembélé: Jacquet Langlands Correspondence to compute Hilbert-Siegel modular forms.
- Faber-van der Geer: compute trace of Hecke operators using analogue of Eichler Shimura
- Currently no modular symbol algorithm

Let F be a nonzero Hecke eigenform. Let

$$L(F, s, \text{spin}) = \prod_{p \text{ prime}} L_p(F, p^{-s})^{-1}$$

where

$$L_p(F, X) = 1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})X^2 - \lambda(p)p^{2k-3}X^3 + p^{4k-6}X^4$$

be the spinor L-function. $L(F, s, \text{spin})$ has an analytic continuation to the whole plane when F is not a Saito-Kurokawa lift. When F is a lift, it has a pole.

Boecherer's Conjecture

- Let $\varepsilon(f)$ be the order the order of the unit group of the quadratic form f and $A(D) := \left(\sum_{f>0,\text{disc } f=D} C(f)/\varepsilon(f)\right)$. Then for negative fundamental discriminants D,

$$L(F, k-1, \chi_D) = C_F|D|^{1-k}A(D)^2$$

  where the LHS is the central critical value of the quadratic twist of $L(F, s)$.
- Explicitly says that Fourier coefficients encode more data than eigenvalues.
- Very recently checked using an optimized variant of our package for weights that have nonrational eigenforms.