LIFTING PUZZLES IN DEGREE 4

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ABSTRACT. We identify the majority of Siegel modular eigenforms in degree 4 and weights up to 16 as being Duke-Imamoḡlu-Ikeda or Miyawaki-Ikeda lifts. We give two examples of eigenforms that are probably also lifts but of an undiscovered type.

1. INTRODUCTION

For degree 4, vector spaces of Siegel modular cusp forms are known through weights $k \leq 16$. For weight 16 there are 7 eigenforms: three quadratic pairs and one rational cusp form. From the work of Ikeda, the L-functions of two of the quadratic pairs are known. In this article, we compute the Euler 2-factors for the remaining quadratic pair and for the rational eigenform. Some of the roots of these Euler factors are not unimodular and so, presumably, there are two new lifts to be discovered. If not, the Generalized Ramanujan-Petersson Conjecture needs significant modification. The unexplained Euler 2-factors do factor in interesting ways, see Table 2. This article surveys all L-functions for degree 4 and weights $k \leq 16$. We both explain how to compute Euler factors in higher degree and actually compute some. We find congruences between particular Euler factors and prove the nonvanishing of particular Miyawaki lifts.

Lifting conjectures have been motivated by the computation of Euler factors of L-functions of Siegel modular Hecke eigenforms. Let S_n^k denote the Siegel modular cusp forms of degree n and weight k. For Hecke eigenforms in S_2^{10} and S_2^{12}, Kurokawa [15] computed some Euler factors and noted their factorization into Euler products of elliptic modular cusp forms and zeta functions.

1. Saito-Kurokawa-Maass Lift *For even weights k, each eigenform $f \in S_1^{2k-2}$ corresponds to an eigenform $F \in S_2^k$ such that the standard L-function of F factors*

$$L(F, s, st) = \zeta(s)L(f, s + k - 1)L(f, s + k - 2).$$

This was proven by Maass, Andrianov and Zagier, compare [5]. For eigenforms in S_3^{12} and S_3^{14}, Miyawaki [16] computed some Euler factors and used their factorizations to give two conjectures. We rephrase them as follows:

2. Miyawaki Conjecture I *For even k, each pair of eigenforms $f \in S_1^{2k-4}$ and $g \in S_1^k$ corresponds to an eigenform $F \in S_3^k$ such that*

$$L(F, s, st) = L(f, s + k - 2)L(f, s + k - 3)L(g, s, st).$$

3. Miyawaki Conjecture II *For even k, each pair of eigenforms $f \in S_1^{2k-2}$ and $g \in S_1^{k-2}$ corresponds to an eigenform $F \in S_3^k$ such that*

$$L(F, s, st) = L(f, s + k - 1)L(f, s + k - 2)L(g, s, st).$$

Duke and Imamoḡlu studied the factorization of the Euler factors of the Schottky form in S_4^8 and, in analogy with the Saito-Kurokawa lift, conjectured a lifting proven by Ikeda [8].

4. Duke-Imamoḡlu-Ikeda Lift *For even k, n and $(n + k)/2$, each eigenform $f \in S_1^k$ corresponds to an eigenform $I_n(f) \in S_n^{(n+k)/2}$ whose standard L-function is*

$$L(I_n(f), s, st) = \zeta(s) \prod_{i=1}^{n} L(f, s + \frac{n+k}{2} - i).$$

Ikeda has also solved Miyawaki's first conjecture with an additional natural non-vanishing condition, see [9]. See also [11] for the resolution of a germane conjecture in [9]. In contrast, Miyawaki's second conjecture remains open. Given a complex function g on the Siegel upper half space, define g^c by $g^c(Z) = \overline{g(-\bar{Z})}$.

5. Miyawaki-Ikeda Lift *Let k, $n + r$ and $\hat{k} = (n + r + k)/2$ be even integers with $r \leq n$. Each pair of eigenforms $f \in S_1^k$ and $g \in S_r^{\hat{k}}$ corresponds to an element $M_n(f, g) \in S_n^{\hat{k}}$ via $M_n(f, g)(Z_n) = \langle I_{n+r}(f)(Z_n \oplus Z_r), g^c(Z_r) \rangle$, where $\langle \cdot, \cdot \rangle$ is the Petersson inner product. If $M_n(f, g)$ is nontrivial then it is an eigenform for the even Hecke algebra and*

$$L(M_n(f, g), s, st) = L(g, s, st) \prod_{i=r+1}^{n} L(f, s + \hat{k} - i).$$

A persistent theme in the articles of Kurokawa and Miyawaki is the difficulty of computing Fourier coefficients and Euler factors of eigenforms. The Fourier coefficients for our examples were computed in [20] and here we use Krieg's matrix [14] to compute Euler factors from Hecke eigenvalues. In particular, we compute Euler 2-factors for all the Hecke eigencuspforms in degree 4 where $\dim S_4^k$ is currently known, namely $k \leq 16$. Two of our examples look suspiciously like unknown lifts because their Euler 2-factors possess rational factors of lower degree and because not all their Satake parameters are unimodular. However, we cannot identify these factors with Euler factors of eigenforms of lower degree and so leave the matter as lifting puzzles.

The most interesting case is weight 16 with $\dim S_4^{16} = 7$, see Table 2. Here there are three pairs of eigenforms with quadratic irrationalities and one rational eigenform. Two eigenforms, h_1 and h_2, are Ikeda lifts. Two eigenforms, h_5 and h_6, are Miyawaki lifts and we prove this by verifying the nonvanishing of the relevant inner product. Another pair, h_3 and h_4, have Euler 2-factors which factor into elliptic factors and a rational quartic factor of unknown origin. Neither do we know what to make of the rational form h_7, whose Euler 2-factor has no unimodular roots.

We summarize the results of our computations: For an elliptic modular eigenform $f \in S_1^k$, we may normalize f so that the Fourier expansion $f(\tau) = \sum_{j=1}^{\infty} a(j; f) q^j$ satisfies $a(1; f) = 1$. The Hecke L-function is then defined by

$$L(f, s) = \prod_{p \text{ prime}} Q_p(f, p^{-s})^{-1}, \text{ where } Q_p(f, X) = 1 - a(p; f)X + p^{k-1}X^2.$$

Let $\phi_{28}^{\pm}(\tau) = q + (-4140 \pm 108\beta)q^2 + \dots$ with $\beta = \sqrt{18209}$ be the two normalized eigenforms of weight 28. The standard L-function of the Ikeda lift $I_4(\phi_{28}^{\pm}) \in S_4^{16}$ is

$$L(I_4(\phi_{28}^{\pm}), s, \text{st}) = \zeta(s)L(\phi_{28}^{\pm}, s+12)L(\phi_{28}^{\pm}, s+13)L(\phi_{28}^{\pm}, s+14)L(\phi_{28}^{\pm}, s+15).$$

The Euler 2-factor is given by

$$(1-X)Q_2(\phi_{28}^{\pm}, 2^{-12}X)Q_2(\phi_{28}^{\pm}, 2^{-13}X)Q_2(\phi_{28}^{\pm}, 2^{-14}X)Q_2(\phi_{28}^{\pm}, 2^{-15}X) =$$
$$(1-X)\left(1 - 2^{-12}(-4140 \pm 108\beta)X + 2^3 X^2\right)\left(1 - 2^{-13}(-4140 \pm 108\beta)X + 2^1 X^2\right)$$
$$\times \left(1 - 2^{-14}(-4140 \pm 108\beta)X + 2^{-1}X^2\right)\left(1 - 2^{-15}(-4140 \pm 108\beta)X + 2^{-3}X^2\right).$$

These are exactly the Euler 2-factors computed for h_1 and h_2 in Table 2, so that h_1 is the Ikeda lift of ϕ_{28}^{+} to degree four and h_2 is the lift of ϕ_{28}^{-}.

To identify h_5 and h_6 as Miyawaki lifts, let $\phi_{26}(\tau) = q - 48q^2 + \dots$ be the normalized eigenform of weight 26 and $\phi_{30}^{\pm}(\tau) = q + (4320 \pm 96\gamma)q^2 + \dots$ with $\gamma = \sqrt{51349}$ be the two normalized eigenforms of weight 30. The two Saito-Kurokawa lifts $I_2(\phi_{30}^{\pm})$ span S_2^{16}. In section 5 we prove that $M_4(\phi_{26}, I_2(\phi_{30}^{\pm}))$ is nontrivial, so that it is an eigenform with standard L-function

$$L(M_4(\phi_{26}, I_2(\phi_{30}^{\pm})), s, \text{st}) = L(I_2(\phi_{30}^{\pm}), s, \text{st})L(\phi_{26}, s+12)L(\phi_{26}, s+13)$$
$$= \zeta(s)L(\phi_{30}^{\pm}, s+14)L(\phi_{30}^{\pm}, s+15)L(\phi_{26}, s+12)L(\phi_{26}, s+13).$$

The Euler 2-factor is given by

$$(1-X)\left(1 - 2^{-14}(4320 \pm 96\gamma)X + 2^1 X^2\right)\left(1 - 2^{-15}(4320 \pm 96\gamma)X + 2^{-1}X^2\right)$$
$$\times \left(1 - 2^{-12}(-48)X + 2^1 X^2\right)\left(1 - 2^{-13}(-48)X + 2^{-1}X^2\right).$$

From Table 2 we see that $M_4(\phi_{26}, I_2(\phi_{30}^{+}))$ gives h_5 and $M_4(\phi_{26}, I_2(\phi_{30}^{-}))$ gives h_6.

Consider the pair h_3 and h_4. From Table 2, the Euler 2-factor of h_3 is given by

$$Q_2(h_3, X, \text{st}) = (1-X)Q_2(\phi_{28}^{+}, 2^{-14}X)Q_2(\phi_{28}^{+}, 2^{-15}X)$$
$$\times \left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right).$$

Thus, h_3 looks like a lift but we do not know the origin of the quartic factor. It is the only factor in Table 2 with unimodular roots, namely $\eta \pm i\sqrt{1-\eta^2}$ for $\eta = (9 \pm \sqrt{1277521})/2048$. Both quadratic factors, $Q_2(\phi_{28}^{+}, 2^{-14}X)$ and $Q_2(\phi_{28}^{+}, 2^{-15}X)$, are shared with the Euler 2-factor of h_1 and the Fourier coefficients of h_1 and h_3 are both algebraic integers in $\mathbb{Q}(\sqrt{18209})$. We will also see that h_1 and h_3 are *congruence neighbors* in the following sense: their Hecke eigenvalues at $p = 2$ are all congruent modulo certain primes in the ring of integers of $\mathbb{Q}(\sqrt{18209})$; namely at 3, at a prime above 5 and at a prime above 7.

Finally, consider the Euler 2-factor of the rational form h_7. If we introduce a pretend Euler factor with an integer parameter ℓ,

$$\mathcal{Q}^{\pm}(X) = 1 - 2^{\ell}(165 \pm 3\sqrt{764242})X + 2^{21+2\ell}X^2,$$

then we have the following factorization for any choice of ℓ:

$$Q_2(h_7, X, \text{st}) = (1-X)\mathcal{Q}^{+}(2^{-10-\ell}X)\mathcal{Q}^{+}(2^{-11-\ell}X)\mathcal{Q}^{-}(2^{-10-\ell}X)\mathcal{Q}^{-}(2^{-11-\ell}X).$$

These look like Euler factors of weight $22 + 2\ell$ with the s variable translated by $10 + \ell$ and $11 + \ell$. However, none of the level one elliptic Euler 2-factors from weights $k = 24$, 28, 30, 32, 34 or 38 have coefficients that generate the quadratic field $\mathbb{Q}(\sqrt{764242})$. We thank S. Hayashida for helpful discussions. We thank B. Heim for pointing out that Miyawaki lifts are currently only known to be eigenforms for the even part of the Hecke algebra. We thank O. King for giving us a nice LISP program to compute the F_p polynomials via Katsurada's recursion formulae.

2. Notation and Background

In this section we give basic definitions and recall theorems of Andrianov and Satake. We define Siegel modular forms, the Hecke algebra, the spherical map and the Satake parameters. For a commutative ring R let $M_{m \times n}(R)$ denote the R-module of m-by-n matrices with coefficients in R. For $x \in M_{m \times n}(R)$ let $x' \in M_{n \times m}(R)$ denote the transpose. Let $V_n(R) = \{x \in M_{n \times n}(R) : x' = x\}$ be the symmetric n-by-n matrices over R; $V_n(\mathbb{R})$ is a euclidean vector space under the inner product $\langle x, y \rangle = \text{tr}(xy)$. For $R \subseteq \mathbb{R}$, an element $x \in V_n(R)$ is called positive definite, written $x > 0$, when $v'xv > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$; we denote the set of these by $\mathcal{P}_n(R)$. The half-integral matrices are $\mathcal{X}_n = \{T \in \mathcal{P}_n(\mathbb{Q}) : \forall v \in \mathbb{Z}^n, v'Tv \in \mathbb{Z}\}$.

Let $\text{GL}_n(R) = \{x \in M_{n \times n}(R) : \det(x) \text{ is a unit in } R\}$ be the general linear group and $\text{SL}_n(R) = \{x \in \text{GL}_n(R) : \det(x) = 1\}$ the special linear group. For $x \in \text{GL}_n(R)$ let x^* denote the inverse transpose. Let $I_n \in \text{GL}_n(R)$ be the identity matrix and set $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \text{SL}_{2n}(R)$. The symplectic group is defined by $\text{Sp}_n(R) = \{x \in \text{GL}_{2n}(R) : x'J_nx = J_n\}$. We write $\Gamma_n = \text{Sp}_n(\mathbb{Z})$ and for $R \subseteq \mathbb{R}$ define the group of positive R-similitudes by $\text{GSp}_n^+(R) = \{x \in M_{2n \times 2n}(R) : \exists \mu \in R^+ : g'J_ng = \mu J_n\}$. Each $\gamma \in \text{GSp}_n^+(R)$ has a unique $\mu = \mu(\gamma) = \det(\gamma)^{1/n}$.

Define the Siegel upper half space $\mathbb{H}_n = \{\Omega \in V_n(\mathbb{C}) : \Im\Omega \in \mathcal{P}_n(\mathbb{R})\}$. The group $\text{GSp}_n^+(\mathbb{R})$ acts on $\mathbb{H}_n$ as $M\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1}$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For any function $f : \mathbb{H}_n \to \mathbb{C}$ and any $k \in \mathbb{Z}$ we follow Andrianov and let $\langle n \rangle = n(n+1)/2$ and define the group action for $M \in \text{GSp}_n^+(R)$ by

$$(f|_k M)(\Omega) = \mu(M)^{kn - \langle n \rangle} \det(C\Omega + D)^{-k} f(M\langle\Omega\rangle).$$

The complex vector space of Siegel modular forms of degree n and weight k is denoted by M_n^k and is defined as the set of holomorphic $f : \mathbb{H}_n \to \mathbb{C}$ such that $f|_k M = f$ for all $M \in \Gamma_n$ and such that for all $Y_0 \in \mathcal{P}_n(\mathbb{R})$, f is bounded on $\{\Omega \in \mathbb{H}_n : \Im\Omega > Y_0\}$. For $f \in M_n^k$ the Siegel Φ-map is defined by $(\Phi f)(\Omega) = \lim_{\lambda \to +\infty} f(\begin{pmatrix} i\lambda & 0 \\ 0 & \Omega \end{pmatrix})$ and the space of cusp forms is defined by $S_n^k = \{f \in M_n^k : \Phi f = 0\}$. Let $e(z) = e^{2\pi i z}$. By the Koecher principle, an $f \in S_n^k$ has a Fourier expansion

$$f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T; f) e\left(\langle T, \Omega \rangle\right)$$

where the $a(T; f)$ satisfy $a(U'TU; f) = \det(U)^k a(T; f)$ for all $U \in \text{GL}_n(\mathbb{Z})$. The *Petersson inner product* $\langle f, g \rangle = \int_{F_n} \det(Y)^{k-n-1} f(Z)\overline{g(Z)} dZ$ makes S_n^k into an inner product space. Here the integral is over a fundamental domain F_n for the action of Γ_n on $\mathbb{H}_n$ and $Z = X + iY \in \mathbb{H}_n$ and $dZ = \bigwedge_{1 \le i \le j \le n} dX_{ij} \wedge dY_{ij}$.

We now introduce various Hecke rings. For a group Γ contained in a semigroup S, the pair (Γ, S) is called a *Hecke pair* if, for all $s \in S$, $\Gamma\backslash\Gamma s\Gamma$ is finite. The free $\mathbb{Q}$-module on the left cosets Γs with $s \in S$ is denoted $L(\Gamma, S)$. A right action of Γ on $L(\Gamma, S)$ is given by $\Gamma s \mapsto \Gamma s\gamma$ for $\gamma \in \Gamma$ and the Γ-invariant subspace is denoted by $H(\Gamma, S)$. If we identify a double coset $\Gamma s\Gamma = \coprod_i \Gamma s_i$ with $\sum_i \Gamma s_i \in L(\Gamma, S)$ then $H(\Gamma, S)$ is generated by double cosets. The binary operation on $H(\Gamma, S) \times L(\Gamma, S)$ given by $(\sum a_i \Gamma s_i)(\Gamma s) = \sum a_i \Gamma s_i s$ is well-defined and restricts to make $H(\Gamma, S)$ an associative ring. We define

$$\mathcal{S}_n = \mathrm{GSp}_n^+(\mathbb{Q}); \qquad \mathcal{H}_n = H(\Gamma_n, \mathcal{S}_n) = \text{ the (global) Hecke algebra,}$$

$$\bar{\mathcal{S}}_n = \mathcal{S}_n \cap M_{2n\times 2n}(\mathbb{Z}); \qquad \bar{\mathcal{H}}_n = H(\Gamma_n, \bar{\mathcal{S}}_n) = \text{ the integral Hecke algebra,}$$

$$\mathcal{S}_{n,p} = \mathcal{S}_n \cap \mathrm{GL}_{2n}(\mathbb{Z}[\tfrac{1}{p}]); \qquad \mathcal{H}_{n,p} = H(\Gamma_n, \mathcal{S}_{n,p}) = \text{ the p-part of $\mathcal{H}_n$,}$$

$$\bar{\mathcal{S}}_{n,p} = \mathcal{S}_{n,p} \cap M_{2n\times 2n}(\mathbb{Z}); \qquad \bar{\mathcal{H}}_{n,p} = H(\Gamma_n, \bar{\mathcal{S}}_{n,p}) = \text{ the p-part of $\bar{\mathcal{H}}_n$.}$$

The Hecke algebra is a commutative ring and acts as a ring of endomorphisms on each M_n^k via:

$$f|_k \sum_i a_i \Gamma_n s_i = \sum_i a_i f|_k s_i,$$

an action that stabilizes S_n^k. The subalgebra generated by the double cosets $\Gamma_n s\Gamma_n$ for $s \in \mathrm{GSp}_n^+(\mathbb{Q})$ with $\mu(s) \in (\mathbb{Q}^\times)^2$ is called the *even part* of the Hecke algebra. Each Hecke operator is self-adjoint with respect to the Petersson inner product and so M_n^k has a basis of simultaneous eigenforms. We know that $\mathcal{H}_n$ is generated by $\cup_p \mathcal{H}_{n,p}$ and that $\mathcal{H}_{n,p}$ is generated by $\bar{\mathcal{H}}_{n,p}$ and $\Gamma_n(pI_{2n})^{-1}$. The p-part of the integral Hecke algebra, $\bar{\mathcal{H}}_{n,p}$, is generated by the double cosets, for $1 \leq i \leq n$:

$$T(p) = \Gamma_n \mathrm{diag}(I_n; pI_n)\Gamma_n \text{ and } T_i(p^2) = \Gamma_n \mathrm{diag}(I_{n-i}, pI_i; p^2 I_{n-i}, pI_i)\Gamma_n \ .$$

We also define $T_0(p^2) = \Gamma_n \mathrm{diag}(I_n; p^2 I_n)\Gamma_n$.

It is no small matter to turn these abstract definitions into a programmable action on the Fourier coefficients. The following formulae in [2] are what we require here. Using the notation in section 2, for

$$f(\Omega) = \sum_{T \in \mathcal{X}_n} a(T; f)e(\langle T, \Omega \rangle) \in S_n^k,$$

the Tth coefficient of $f|T(p)$ (respectively, $f|T_i(p^2)$) is given by

$$a(T; f|T(p)) = \sum_{\substack{0 \leq r \leq n \\ U \in G(r, n-r, 0)}} p^{\frac{r(r-2n+2k-1)}{2}} a(\tfrac{1}{p}T[UD(r, n-r, 0)]; f)$$

and

$$a(T; f|T_i(p^2)) = \sum_{\substack{0 \leq r,s \leq n \\ r+s \leq n \\ U \in G(r,s,n-r-s)}} p^{(2r+s)(k-n+r-1)-r(r-1)} e(T, U)$$

$$\times a(\tfrac{1}{p^2}T[UD(r, s, n-r-s)]; f).$$

In the above formulas, use the following notation: Define the exponential sum

$$e(T,U) = \sum_{\substack{M \bmod p \\ \operatorname{rank}_p(M) = s-(n-i)}} e\left(\frac{1}{p}\operatorname{tr}(T[U]\operatorname{diag}(0^r, M^s, 0^{n-r-s}))\right)$$

where the notation M^s indicates that $M \in M_{s\times s}(\mathbb{Q})$ and 0 indicates the zero matrix. The summation is only over symmetric M.

The representatives of double cosets are found by enumerating the set $G(r,s,t)$ for $r+s+t=n$, which is the union of the following (extend the notation to let $M^{r,s}$ denote an element of $M_{r\times s}(\mathbb{Q})$):

$$\left\{ \left(\begin{smallmatrix} x & 1 \\ U & 0 \end{smallmatrix}\right) : U^{(n-1)} \in G(r,s,t-1), x = (pa,0), a^{1,r} \bmod p \right\}$$

$$\left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ x & I \end{smallmatrix}\right) : U^{n-1} \in G(r-1,s,t), x = (0,a,b)', a^{1,s} \bmod p, b^{1,t} \bmod p^2 \right\}$$

and

$$\left\{ \left(\begin{smallmatrix} 0 & 1 & 0 \\ U_1 & 0 & U_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & I \end{smallmatrix}\right) : (U_1^{n-1,r}, U_2^{n-1,n-r-1}) \in G(r,s-1,t), x = (0,a)', a^{1,t} \bmod p \right\}$$

where it is understood that $G(r,s,t) = \emptyset$ if any of r,s or t is negative. The only comment we add to these formulae of Breulmann and Kuss is that, in the summation for $e(T,U)$, a zero dimensional matrix of rank 0 must be allowed; that is, we take $e(T,U) = 1$ when $i = n$ and $s = 0$. These formulae were used to compute the Hecke eigenvalues in Table 1 from the Fourier coefficients at [18].

By a fundamental result of Satake, the p-part of the integral Hecke algebra is isomorphic to the invariant ring $\mathbb{Q}[x_0, \ldots, x_n]^{W_n}$, where W_n is the group generated by the permutations of $x_1, \ldots, x_n$ and by the automorphisms

$$\tau_i(x_0) = x_0 x_i, \ \tau_i(x_i) = x_i^{-1}, \ \tau_j(x_i) = x_i \text{ for } i \neq j.$$

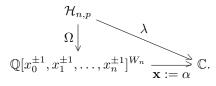To define this isomorphism, write left cosets from $\Gamma_n \backslash \mathcal{S}_{n,p}$ in the form $\Gamma_n M$ with

$$\mu(M) = p^\delta \text{ and } M = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \text{ and } D = \begin{pmatrix} p^{\delta_1} & & & * \\ & p^{\delta_2} & & \\ & & \ddots & \\ 0 & & & p^{\delta_n} \end{pmatrix}.$$

The terms p^δ and $\mathrm{GL}_n(\mathbb{Z})D$ depend only on the left coset $\Gamma_n M$ and the sequence of elementary divisors p^{δ_i} is determined by $\mathrm{GL}_n(\mathbb{Z})D$. Letting

$$\Omega(\Gamma_n M) = x_0^\delta \prod_{i=1}^n \left(\frac{x_i}{p^i}\right)^{\delta_i}$$

defines an algebra homomorphism $\Omega : L(\Gamma_n, \mathcal{S}_{n,p}) \to \mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ that induces the Satake isomorphism $\Omega : \bar{\mathcal{H}}_{n,p} \to \mathbb{Q}[x_0, \ldots, x_n]^{W_n}$. The map Ω is a universal object for complex one dimensional representations of $\mathcal{H}_{n,p}$, see page 165 of [1].

Theorem 2.1. *Let $\lambda \in \operatorname{Hom}_{\mathbb{Q}}(\mathcal{H}_{n,p}, \mathbb{C})$ be a nontrivial homomorphism. There is an $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$ such that the following diagram commutes:*

$$\begin{array}{ccc} & \mathcal{H}_{n,p} & \\ \Omega \downarrow & \searrow^{\lambda} & \\ \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} & \underset{\mathbf{x} := \alpha}{\longrightarrow} & \mathbb{C}. \end{array}$$

In particular, given an eigenform $F \in S_n^k$ of the Hecke algebra $\mathcal{H}_n$, we may define a homomorphism $\lambda_F : \mathcal{H}_n \to \mathbb{C}$ by $\lambda_F(T)F = F|_k T$. For each prime p, we have the restriction $\lambda_F : \mathcal{H}_{n,p} \to \mathbb{C}$ and the $(\alpha_{0,p}, \ldots, \alpha_{n,p})$ promised in Theorem 2.1 are called the *Satake p-parameters of* F and satisfy $\Omega(T)|_{\mathbf{x}:=\alpha} = \lambda_F(T)$ for $T \in \mathcal{H}_{n,p}$. The Satake parameters of F are used in the definition of the L-functions attached to F. In particular, the standard L-function is given by

$$L(F, s, \mathrm{st}) = \prod_{p \text{ prime}} Q_p(\alpha_p, p^{-s})^{-1}$$

where

$$Q_p(F, X, \mathrm{st}) = Q_p(\alpha_p, X) = (1 - X) \prod_{i=1}^{n} (1 - \alpha_{i,p} X)(1 - \alpha_{i,p}^{-1} X).$$

The spinor L-function is given by

$$L(F, s, \mathrm{spin}) = \prod_{p \text{ prime}} Q_p(F, p^{-s}, \mathrm{spin})^{-1}$$

where

$$Q_p(F, X, \mathrm{spin}) = (1 - \alpha_{0,p} X) \prod_{r=1}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n}^{\binom{n}{r}} (1 - \alpha_{0,p} \alpha_{i_1,p} \ldots \alpha_{i_r,p} X).$$

3. Computing Euler Factors

In recent work the first and third authors have computed bases for the spaces of degree 4 Hecke eigenforms in weights up to 16. In what follows, we show how to compute the Hecke eigenvalues of these forms and from this deduce their Satake parameters at the prime 2. In all cases, the standard L-functions factor, suggesting that all the forms in [20] are lifts. All but three of the computed forms we identify as either Ikeda lifts or Miyawaki lifts, leaving us wondering if there are two previously unidentified lifts into genus 4.

In what follows, we make use of the following notation. For a polynomial $f \in \mathbb{Q}[x_0, \ldots, x_n]$ we set $(\sigma \cdot f)(x_0, \ldots, x_n) = f(\sigma(x_0), \ldots, \sigma(x_n))$ and set

$$[f] = \sum_{\sigma \in W_n / \mathrm{Stab}(f)} \sigma \cdot f.$$

Also, if $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, by $x^{\mathbf{b}}$ we mean the monomial $x_1^{b_1} \cdots x_n^{b_n}$. Similar meaning is implied by $\alpha^{\mathbf{b}}$.

Parallel to the generators for $\bar{\mathcal{H}}_{n,p}$, the ring $\mathbb{Q}[x_0, x_1, \ldots, x_n]^{W_n}$ is generated by $x_0(1+x_1) \ldots (1+x_n)$ and $[x_0^2 x^{(1,1,\ldots,1)}], [x_0^2 x^{(2,1,\ldots,1)}], \ldots, [x_0^2 x^{(2,2,\ldots,2,1)}]$. Note that, if there are ℓ twos in the superscript $(2, \ldots, 2, 1, \ldots, 1)$, we have

$$[x^{(2,\ldots,2,1,\ldots,1)}] = x_0^2 x_1 \ldots x_n \, V_\ell(\mathbf{x}) \text{ where}$$

$$V_\ell(\mathbf{x}) = \sum_{e_1,\ldots e_n \in \{1,0,-1\}:|e_1|+\cdots+|e_n|=\ell} x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}.$$

We note that $V_\ell(x)$ is just the ℓth elementary symmetric polynomial in the variables $x_i + x_i^{-1}$ and we set

$$V(\mathbf{x}) = (V_0(\mathbf{x}), V_1(\mathbf{x}), \ldots, V_n(\mathbf{x})).$$

The generators of $\bar{\mathcal{H}}_{n,p}$ are related to the generators of $\mathbb{Q}[x_0, x_1, \ldots, x_n]^{W_n}$ in the following Theorem [6], see [21] for a more general statement.

Theorem 3.1. *Let n be a positive integer and p a prime. We have*

$$\Omega(T(p)) = x_0(1 + x_1) \ldots (1 + x_n).$$

Also, there exists an upper triangular matrix $K_n(p^2) \in M_{(n+1)\times(n+1)}(\mathbb{Z}[\frac{1}{p}])$ with positive coefficients on and above the diagonal such that

$$\Omega\left(T_n(p^2), \ldots, T_0(p^2)\right) = \left([x_0^2 x^{(1,1,\ldots,1)}], [x_0^2 x^{(2,1,\ldots,1)}], \ldots, [x_0^2 x^{(2,2,\ldots,2,2)}]\right) K_n(p^2).$$

The key computational point is that, for $n \geq 2$, the entries of the matrix $K_n(p^2)$ have been given by Krieg [14]. For $n = 4$ the Krieg matrix is given by

$$\begin{pmatrix} \frac{1}{p^{10}} & \frac{p^4-1}{p^{10}} & \frac{4p^{17}-p^{16}-2p^{14}-p^{12}}{p^{20}} & \frac{(p^2-1)(4p^{17}-p^{16}-2p^{14}-p^{12})}{p^{20}} & \frac{6p^4-8p^3+2}{p^4} \\ 0 & \frac{1}{p^6} & \frac{p^3-1}{p^6} & \frac{3p^{11}-p^8(1+p+p^2)}{p^{12}} & \frac{(p-1)(3p^3-p^2-p-1)}{p^4} \\ 0 & 0 & \frac{1}{p^3} & \frac{p^2-1}{p^3} & \frac{2(p-1)}{p} \\ 0 & 0 & 0 & \frac{1}{p} & \frac{p-1}{p} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For general n, a Maple program that computes $K_n(p^2)$ may be found at [19].

Definition 3.1. *For $n \in \mathbb{Z}^+$, define $P_n \in M_{(n+1)\times(n+1)}(\mathbb{Z})$ by: $(P_n)_{ij} = 0$ if $i > j$ or $i + j$ is odd and by $(P_n)_{ij} = \binom{n+1-i}{(j-i)/2}$ if $i + j$ is even.*

For example, in $n = 4$,

$$P_4 = \begin{pmatrix} 1 & 0 & 4 & 0 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.2. *Let $n \geq 2$ and k be positive integers and p a prime. Let $F \in M_n^k$ be a simultaneous eigenform of $\bar{\mathcal{H}}_{n,p}$ with Satake p-parameters $(\alpha_0, \ldots, \alpha_n)$. Write the standard Euler p-factor as*

$$Q_p(\mathbf{x}, X) \quad = (1 - X) \sum_{j=0}^{2n} (-1)^j k_j(\mathbf{x}) X^j \text{ with } k_j(\mathbf{x}) = k_{2n-j}(\mathbf{x}).$$

If we write

$$k(\mathbf{x}) = \quad (k_0(\mathbf{x}), k_1(\mathbf{x}), \ldots, k_n(\mathbf{x})),$$
$$\lambda_F(p^2) = \quad \lambda_F\left(T_n(p^2), T_{n-1}(p^2), \ldots, T_0(p^2)\right),$$

then the following relations give $k(\alpha)$ as a linear function of $\lambda_F(p^2)$:

$$k(\alpha) = V(\alpha)P_n \text{ and } \lambda_F(p^2) = p^{nk-\langle n \rangle} V(\alpha)K_n(p^2).$$

Proof. We first note the effect of the slash normalization upon the Satake parameters. For any modular form $F \in M_n^k$ we have

$$F|_k T_n(p^2) = F|_k p\Gamma_n = (p^2)^{nk-\langle n \rangle} \det(pI_n)^{-k} F = p^{nk-2\langle n \rangle} F.$$

Thus the equality $\lambda_F(T_n(p^2)) = p^{nk-2\langle n \rangle}$ is independent of F. Since $T_n(p^2) = p\Gamma_n$ consists of one coset, we may compute $\Omega(T_n(p^2)) = p^{-\langle n \rangle} x_0^2 x_1 \ldots x_n$ directly

from the definition. Therefore, the Satake parameters of an eigenform F satisfy $\alpha_0^2 \alpha_1 \ldots \alpha_n = p^{nk - \langle n \rangle}$.

In the defining relation of the Krieg matrix, $K_n(p^2)$,

$$\Omega\left(T_n(p^2), \ldots, T_0(p^2)\right) = \left([x_0^2 x^{(1,1,\ldots,1)}], [x_0^2 x^{(2,1,\ldots,1)}], \ldots, [x_0^2 x^{(2,2,\ldots,2,2)}]\right) K_n(p^2),$$

we may substitute the Satake parameters $\mathbf{x} := \alpha$ to obtain

$$\lambda_F(p^2) = p^{nk - \langle n \rangle} V(\alpha) K_n(p^2).$$

The other relation, $k(\mathbf{x}) = V(\mathbf{x}) P_n$, is combinatorial; for example, when ℓ is even, the coefficient of X^ℓ in $\prod_{i=1}^n (1 - (x_i + x_i^{-1})X + X^2)$ is

$$V_\ell(\mathbf{x}) + \binom{n - (\ell - 2)}{1} V_{\ell-2}(\mathbf{x}) + \binom{n - (\ell - 4)}{2} V_{\ell-4}(\mathbf{x}) + \ldots$$

$\square$

With these formulae from the Krieg matrix we may compute the Euler 2-factors in Table 2 from the Hecke eigenvalues in Table 1. As a check on computations, one may use the following relation.

Theorem 3.3. *For $n \geq 2$, the following relation holds in $\bar{\mathcal{H}}_{n,p}$:*

$$T(p)^2 = T_0(p^2) + (1+p)T_1(p^2) + (1+p)(1+p^2)T_2(p^2) + \cdots + (1+p)\ldots(1+p^n)T_n(p^2).$$

Proof. This is Proposition 5.1 in Hafner-Walling [7], rewritten in Andrianov's notation. $\square$

4. Computing Duke-Imamoḡlu-Ikeda Lifts

Let n, k and $(n+k)/2$ be even. Let $\phi \in S_1^k$ be an eigenform. The Ikeda lift of ϕ to $I_n(\phi) \in S_n^{(n+k)/2}$ is described in [8]. Let $\psi \in S_1^{(k+1)/2}(\Gamma_0(4))^+$ be a form corresponding to ϕ under the Shimura correspondence. (In this paper we always normalize ψ so that the leading coefficient of its Fourier expansion is 1 and thereby give meaning to equations like $I_4(\phi_{16}) = -168\, G_{10}$. This somewhat arbitrary choice aids the reproduction of our results and the debugging of code.)

For a fixed $T \in \mathcal{X}_n$, here is how to calculate the Fourier coefficient $a(T; I_n(\phi))$. For each prime p that divides $\det(2T)$, along with the prime 2, calculate the genus symbol G_p of T, see [3]. Using the recursion worked out by Katsurada [10], the polynomial F_p corresponding to this genus symbol is computed. We refer the reader to [8] or [10] for the definition of the F_p polynomial. We thank O. King for giving us his LISP program that computes F_p when the genus symbol G_p is given as input, compare [13].

Use the rational function

$$\tilde{F}_p(x) = x^{-(\deg F_p)/2} F_p(p^{-(n+1)/2} x)$$

with the property that $\tilde{F}_p(x^{-1}) = \tilde{F}_p(x)$. Define

$$b_p = p^{-(k-1)/2}\left(a(p; \phi) + \sqrt{a(p; \phi)^2 - 4p^{k-1}}\right)/2$$

so that we have $(1 - p^{(k-1)/2}b_p x)(1 - p^{(k-1)/2}b_p^{-1} x) = 1 - a(p; \phi)x + p^{k-1}x^2$. Let

$$c_T = \prod_{\text{primes } p:\ p|\det(2T) \text{ or } p=2} \tilde{F}_p(b_p).$$

Define a function $\delta : \mathbb{Z} \setminus \{0\} \to \mathbb{Z} \setminus \{0\}$ by

$$
\delta(y) = \begin{cases} y & \text{if } y \text{ is square free and } y \equiv 1 \mod 4 \\ 4y & \text{if } y \text{ is square free and } y \not\equiv 1 \mod 4 \\ \delta(\bar{y}) & \text{where } \bar{y} \text{ is square free and } y = \ell^2 \bar{y} \text{ for some } \ell \in \mathbb{Z}. \end{cases}
$$

The Ikeda Fourier coefficient at T is

$$
a(T; I_n(\phi)) = a(|\delta((-1)^{n/2} \det(2T))|; \psi) \left(\frac{\det(2T)}{|\delta((-1)^{n/2} \det(2T))|} \right)^{(k-1)/4} c_T.
$$

5. Computing Miyawaki-Ikeda Lifts

The Witt maps are given by restriction to the reducible locus, see [22].

Proposition 5.1. *For $i + j = n$ there are homomorphisms $\psi_{ij}^* : S_n^k \to S_i^k \otimes S_j^k$ for $\psi_{ij} : \mathbb{H}_i \times \mathbb{H}_j \to \mathbb{H}_n$ defined by $\psi_{ij}(\Omega_1, \Omega_2) = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$. Furthermore, if $n = 2i$ we have $\psi_{ii}^* : S_n^k \to \mathrm{Sym}\left(S_i^k \otimes S_i^k \right)$.*

The following is essentially due to Ozeki, compare [17].

Proposition 5.2. *For all $f \in S_n^k$ we have*

$$
a\left(T_1 \otimes T_2; \psi_{ij}^* f \right) = \sum_{T \in \mathcal{X}_n : \, \pi_{i \times i}^{upper}(T) = T_1, \, \pi_{j \times j}^{lower}(T) = T_2} a(T; f).
$$

The Fourier coefficients of Witt images are typically calculated by the enumeration of T with fixed T_1 and T_2; see page 210 of [20] for a specific example.

In the next Proposition we use the generators of the ring $M_2 = \mathbb{C}[E_4, E_6, \chi_{10}, \psi_{12}]$ given by Igusa. We refer to [20] for the definition of the various eigenforms f_i and for the normalization of Fourier coefficients as well. We let root lattices A_4, D_4, etc., stand for any Gram matrix of the corresponding lattice, see [3].

Proposition 5.3. *In $S_4^{14} \otimes S_2^{14}$ we have*

$$
\psi_{42}^* I_6(\phi_{22}) = 1848 \, f_9 \otimes \chi_{10} E_4.
$$

Here f_9 is the element of a Hecke eigenbasis that is not an Ikeda lift, see [20].

Proof. The space S_2^{14} is one dimensional, spanned by $\chi_{10} E_4$ and determined by the single Fourier coefficient $a(\frac{1}{2} A_2)$. The space S_4^{14} is three dimensional, spanned by f_9 and two Ikeda lifts $I_4(\phi_{24}^{\pm})$ and determined by the three Fourier coefficients $a(\frac{1}{2} D_4)$, $a(\frac{1}{2} A_4)$ and $a(\frac{1}{2}(A_1 \oplus A_3))$, see page 214 in [20]. Hence, the element $\psi_{42}^* I_6(\phi_{22})$ of $S_4^{14} \otimes S_2^{14}$ is determined by the first three of the following four Fourier coefficients. The fourth coefficient is redundant but serves as a check. Computing

as in Proposition 5.2 we have

$$a\left(\tfrac{1}{2}D_4 \otimes \tfrac{1}{2}A_2; \psi_{42}^* I_6(\phi_{22})\right) =$$
$$a(\tfrac{1}{2}D_4 \oplus \tfrac{1}{2}A_2; I_6(\phi_{22})) + 72a(\tfrac{1}{2}D_6; I_6(\phi_{22})) + 192a(\tfrac{1}{2}E_6; I_6(\phi_{22}))$$
$$= 2784 + 72(10) + 192(1) = 3696 = 1848(2),$$

$$a\left(\tfrac{1}{2}A_4 \otimes \tfrac{1}{2}A_2; \psi_{42}^* I_6(\phi_{22})\right) =$$
$$a(\tfrac{1}{2}A_4 \oplus \tfrac{1}{2}A_2; I_6(\phi_{22})) + 120a(\tfrac{1}{2}D_6; I_6(\phi_{22})) + 240a(\tfrac{1}{2}E_6; I_6(\phi_{22}))$$
$$+ 30a(\tfrac{1}{2}A_6; I_6(\phi_{22}))$$
$$= -8040 + 120(10) + 240(1) + 30(-88) = -9240 = 1848(-5),$$

$$a\left(\tfrac{1}{2}(A_2 \oplus A_2) \otimes \tfrac{1}{2}A_2; \psi_{42}^* I_6(\phi_{22})\right) =$$
$$a(\tfrac{1}{2}(A_2 \oplus A_2 \oplus A_2); I_6(\phi_{22})) + 432a(\tfrac{1}{2}D_6; I_6(\phi_{22})) + 252a(\tfrac{1}{2}E_6; I_6(\phi_{22}))$$
$$+ 216a(\tfrac{1}{2}A_6; I_6(\phi_{22})) + 24a(\tfrac{1}{2}D_4 \oplus \tfrac{1}{2}A_2; I_6(\phi_{22})) + 36a(\tfrac{1}{2}A_4 \oplus \tfrac{1}{2}A_2; I_6(\phi_{22}))$$
$$= -128844 + 432(10) + 252(1) + 216(-88) + 24(2784) + 36(-8040)$$
$$= -365904 = 1848(-198).$$

$$a\left(\tfrac{1}{2}(A_1 \oplus A_3) \otimes \tfrac{1}{2}A_2; \psi_{42}^* I_6(\phi_{22})\right) =$$
$$a(\tfrac{1}{2}(A_1 \oplus A_3 \oplus A_2); I_6(\phi_{22})) + 264a(\tfrac{1}{2}D_6; I_6(\phi_{22})) + 288a(\tfrac{1}{2}E_6; I_6(\phi_{22}))$$
$$+ 96a(\tfrac{1}{2}A_6; I_6(\phi_{22})) + 90a(\tfrac{1}{2}D_5 \oplus \tfrac{1}{2}A_1; I_6(\phi_{22})) + 24a(\tfrac{1}{2}A_5 \oplus \tfrac{1}{2}A_1; I_6(\phi_{22}))$$
$$+ 6a(\tfrac{1}{2}A_3 \oplus \tfrac{1}{2}A_3; I_6(\phi_{22}))$$
$$= -54120 + 264(10) + 288(1) + 96(-88) + 90(-132) + 24(736) + 6(17600)$$
$$= 51744 = 1848(28).$$

The Fourier coefficients $a(\cdot; I_6(\phi_{22}))$ are computed according to section 4. Since $a(\tfrac{1}{2}D_4; f_9) = 2$, $a(\tfrac{1}{2}A_4; f_9) = -5$, $a(\tfrac{1}{2}(A_2 \oplus A_2); f_9) = -198$ and $a(\tfrac{1}{2}(A_1 \oplus A_3); f_9) = 28$, see Table 2 on page 218 of [20], and $a(\tfrac{1}{2}A_2; \chi_{10}E_4) = 1$ we have the result. $\square$

Corollary 5.4. *The eigenform $f_9 \in S_4^{14}$ is a Miyawaki lift, namely $M_4(\phi_{22}, \chi_{10}E_4)$ $= 1848\langle \chi_{10}E_4, \chi_{10}E_4\rangle f_9$.*

Proof. By definition, we have $M_4(\phi_{22}, \chi_{10}E_4) = \langle \psi_{42}^* I_6(\phi_{22}), (\chi_{10}E_4)^c\rangle$. Since $\chi_{10}E_4$ has Fourier coefficients in $\mathbb{Q}$, a totally real field, we have $(\chi_{10}E_4)^c = \chi_{10}E_4$. By Proposition 5.3, we conclude $\langle \psi_{42}^* I_6(\phi_{22}), (\chi_{10}E_4)^c\rangle = \langle 1848\, f_9 \otimes \chi_{10}E_4, \chi_{10}E_4\rangle$ $= 1848\,\langle \chi_{10}E_4, \chi_{10}E_4\rangle f_9$. $\square$

A similar but more involved computation gives the following Proposition and Corollary. Since S_2^{16} is two dimensional, we expect to get two Miyawaki lifts from the Ikeda lift $I_6(\phi_{26}) \in S_6^{16}$.

Proposition 5.5. *Let $\omega^\pm = 48(1643168 \pm 2141\gamma)/200620543$ for $\gamma = \sqrt{51349}$. In $S_4^{16} \otimes S_2^{16}$ we have*

$$\psi_{42}^* I_6(\phi_{26}) = \omega^- h_5 \otimes I_2(\phi_{30}^+) + \omega^+ h_6 \otimes I_2(\phi_{30}^-).$$

Corollary 5.6. *The eigenforms $h_5, h_6 \in S_4^{16}$ are Miyawaki lifts, $M_4(\phi_{26}, I_2(\phi_{30}^+))$ $= \omega^-\langle I_2(\phi_{30}^+), I_2(\phi_{30}^+)\rangle h_5$ and $M_4(\phi_{26}, I_2(\phi_{30}^-)) = \omega^+\langle I_2(\phi_{30}^-), I_2(\phi_{30}^-)\rangle h_6$.*

6. Final Remarks

Hecke eigenbases of S_4^k for even $k \leq 16$ were given in [20]. For odd $k \leq 16$, the S_4^k are trivial, see [4] for some of these results. We survey these standard L-functions in degree 4. The Schottky form J_8 spans S_4^8 and $I_4(J_8) = -120\, J_8$ so that

$$L(J_8, s, \mathrm{st}) = \zeta(s)L(\Delta_{12}, s+4)L(\Delta_{12}, s+5)L(\Delta_{12}, s+6)L(\Delta_{12}, s+7).$$

The form G_{10} spans S_4^{10} and $I_4(\phi_{16}) = -168\, G_{10}$ so that

$$L(G_{10}, s, \mathrm{st}) = \zeta(s)L(\phi_{16}, s+6)L(\phi_{16}, s+7)L(\phi_{16}, s+8)L(\phi_{16}, s+9).$$

The space $S_4^{12} = \mathbb{C}f_5 + \mathbb{C}f_6$ has $M_4(\phi_{18}, I_2(\phi_{22}))/\langle I_2(\phi_{22}), I_2(\phi_{22})\rangle = 144f_6$ and $I_4(\phi_{20}) = 360\, f_5$. As given by Ikeda [9]:

$$L(f_5, s, \mathrm{st}) = \quad \zeta(s)L(\phi_{20}, s+8)L(\phi_{20}, s+9)L(\phi_{20}, s+10)L(\phi_{20}, s+11),$$
$$L(f_6, s, \mathrm{st}) = \quad \zeta(s)L(\phi_{18}, s+8)L(\phi_{18}, s+9)L(\phi_{22}, s+10)L(\phi_{22}, s+11).$$

The space $S_4^{14} = \mathbb{C}f_7 + \mathbb{C}f_8 + \mathbb{C}f_9$ has $M_4(\phi_{22}, I_2(\phi_{26}))/\langle I_2(\phi_{26}), I_2(\phi_{26})\rangle = 1848\, f_9$, and the two Ikeda lifts $I_4(\phi_{24}^+) = 12\, f_7$ and $I_4(\phi_{24}^-) = 12\, f_8$.

$$L(f_7, s, \mathrm{st}) = \quad \zeta(s)L(\phi_{24}^+, s+10)L(\phi_{24}^+, s+11)L(\phi_{24}^+, s+12)L(\phi_{24}^+, s+13),$$
$$L(f_8, s, \mathrm{st}) = \quad \zeta(s)L(\phi_{24}^-, s+10)L(\phi_{24}^-, s+11)L(\phi_{24}^-, s+12)L(\phi_{24}^-, s+13),$$
$$L(f_9, s, \mathrm{st}) = \quad \zeta(s)L(\phi_{22}, s+10)L(\phi_{22}, s+11)L(\phi_{26}, s+12)L(\phi_{26}, s+13).$$

The L-functions for weight 16, in so far as they are known, were given in the Introduction. Recall that the Euler 2-factor of h_3 is given by

$$(1 - X)Q_2(\phi_{28}^+, 2^{-14}X)Q_2(\phi_{28}^+, 2^{-15}X)\left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right).$$

In the hope of eventually locating this quartic among Euler factors of lower degree, we also give the spinor Euler factor. If $(1-X)\left(1 + \frac{9}{512}X + \frac{1601}{2048}X^2 + \frac{9}{512}X^3 + X^4\right)$ were the standard Euler 2-factor of a degree 2 eigenform of weight k, then the spinor Euler 2-factor would be

$$1 \pm 2^{k-7} \cdot 3 \cdot 5^2 X + 2^{2k-12} \cdot 5 \cdot 7 \cdot 29X^2 \pm 2^{3k-10} \cdot 3 \cdot 5^2 X^3 + 2^{4k-6}X^4 =$$
$$\left(1 \pm 2^{k-8} \cdot 3(5^2 \pm \sqrt{641})X + 2^{2k-3}X^2\right)\left(1 \pm 2^{k-8} \cdot 3(5^2 \mp \sqrt{641})X + 2^{2k-3}X^2\right).$$

Finally, Katsurada has shown that Ikeda lifts have non-Ikeda lifts as congruence neighbors when numerators of certain normalized values of L-functions are divisible by large primes. In our examples, we may verify the following: Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\beta)$. The eigenvalues of the Ikeda lift h_1 and the non-Ikeda lift h_3 have differences $\lambda_{h_3}(T) - \lambda_{h_1}(T) \in 3\mathcal{O} \cap a_5\mathcal{O} \cap a_7\mathcal{O}$ for $T = T(2)$, $T_0(4)$, $T_1(4)$, $T_2(4)$ and $T_3(4)$, where a_5 of norm 5 and a_7 of norm 7 are given, respectively, by 356429818034565355925181170153310006194938089260129956619938561127352927 3539155883991421 $+$ 264138058267103580115469693640515149001366638840289345 90117770380976171799102958519902β and 327059457215772014369441101017343 40267987768965840379 9880722335340075861248530834331750871 $6 +$ 24237267926 469934357153909329439375246810596424836893715023944129121867151461567374 501869β. We thank H. Katsurada for bringing his work [12] to our attention.

TABLE 1. Hecke eigenvalues for all 7 eigenforms in S_4^{16} as given in [20] where $\beta = \sqrt{18209}$ and $\gamma = \sqrt{51349}$.

h_1	$T(2)$	$12960(67989 + 443\beta)$
	$T_0(4)$	$276480(2124076963591 + 2858223465\beta)$
	$T_1(4)$	$353894400(3255937 + 7494543\beta)$
	$T_2(4)$	$-42278584320(-1232945 + 801\beta)$
	$T_3(4)$	$515396075520(-523 + 27\beta)$
h_2	$T(2)$	$-12960(-67989 + 443\beta)$
	$T_0(4)$	$-276480(-2124076963591 + 2858223465\beta)$
	$T_1(4)$	$-353894400(-3255937 + 7494543\beta)$
	$T_2(4)$	$42278584320(1232945 + 801\beta)$
	$T_3(4)$	$-515396075520(523 + 27\beta)$
h_3	$T(2)$	$-230400(1703 + 9\beta)$
	$T_0(4)$	$283115520(3556625 + 644607\beta)$
	$T_1(4)$	$1698693120(12749123 + 67821\beta)$
	$T_2(4)$	$43486543872(183579 + 533\beta)$
	$T_3(4)$	$206158430208(221 + 27\beta)$
h_4	$T(2)$	$230400(-1703 + 9\beta)$
	$T_0(4)$	$-283115520(-3556625 + 644607\beta)$
	$T_1(4)$	$-1698693120(-12749123 + 67821\beta)$
	$T_2(4)$	$-43486543872(-183579 + 533\beta)$
	$T_3(4)$	$-206158430208(-221 + 27\beta)$
h_5	$T(2)$	$1175040(557 + \gamma)$
	$T_0(4)$	$70778880(1331260505 + 3483163\gamma)$
	$T_1(4)$	$18119393280(2397057 + 11974\gamma)$
	$T_2(4)$	$38654705664(317997 + 529\gamma)$
	$T_3(4)$	$824633720832(449 + 3\gamma)$
h_6	$T(2)$	$-1175040(-557 + \gamma)$
	$T_0(4)$	$-70778880(-1331260505 + 3483163\gamma)$
	$T_1(4)$	$-18119393280(-2397057 + 11974\gamma)$
	$T_2(4)$	$-38654705664(-317997 + 529\gamma)$
	$T_3(4)$	$-824633720832(-449 + 3\gamma)$
h_7	$T(2)$	230400000
	$T_0(4)$	163381183072174080
	$T_1(4)$	-29115328285900800
	$T_2(4)$	-7821199870525440
	$T_3(4)$	399947354603520

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TABLE 2. Euler 2-factors $Q_2(h_i, X, \mathrm{st})/(1-X)$ for the eigenforms h_i in Table 1, where $\beta = \sqrt{18209}$ and $\gamma = \sqrt{51349}$.

h_1	$2^{-46}\left(-1024 + (-1035 + 27\beta)x - 8192x^2\right)$ $\left(-2048 + (-1035 + 27\beta)x - 4096x^2\right)$ $\left(-4096 + (-1035 + 27\beta)x - 2048x^2\right)$ $\left(-8192 + (-1035 + 27\beta)x - 1024x^2\right)$
h_2	$2^{-46}\left(1024 + (1035 + 27\beta)x + 8192x^2\right)$ $\left(2048 + (1035 + 27\beta)x + 4096x^2\right)$ $\left(4096 + (1035 + 27\beta)x + 2048x^2\right)$ $\left(8192 + (1035 + 27\beta)x + 1024x^2\right)$
h_3	$2^{-34}\left(-2048 + (-1035 + 27\beta)x - 4096x^2\right)$ $\left(-4096 + (-1035 + 27\beta)x - 2048^2\right)$ $\left(2048 + 36x + 1601x^2 + 36x^3 + 2048x^4\right)$
h_4	$2^{-34}\left(2048 + (1035 + 27\beta)x + 4096x^2\right)$ $\left(4096 + (1035 + 27\beta)x + 2048^2\right)$ $\left(2048 + 36x + 1601x^2 + 36x^3 + 2048x^4\right)$
h_5	$2^{-36}\left(-512 + (135 + 3\gamma)x - 1024x^2\right)$ $\left(-1024 + (135 + 3\gamma)x - 512x^2\right)$ $\left(512 + 3x + 256x^2\right)$ $\left(256 + 3x + 512x^2\right)$
h_6	$2^{-36}\left(512 + (-135 + 3\gamma)x + 1024x^2\right)$ $\left(1024 + (-135 + 3\gamma)x + 512x^2\right)$ $\left(512 + 3x + 256x^2\right)$ $\left(256 + 3x + 512x^2\right)$
h_7	$2^{-28}\left(32768 - 5280x - 20755x^2 - 2640x^3 + 8192x^4\right)$ $\left(8192 - 2640x - 20755x^2 - 5280x^3 + 32768x^4\right)$

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