Geometric and Spectral Properties of Compact Riemann Orbisurfaces

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Abstract

Historically, inverse spectral theory has been concerned with the relationship between the geometry and the spectrum of compact Riemannian manifolds, where "spectrum" means the eigenvalue spectrum of the Laplace operator as it acts on smooth functions on a manifold M. We examine this relationship in the non-smooth setting, that is, for manifolds which have singularities. We generalize several relevant geometric tools to the setting of cone-surfaces; in joint work with Hugo Parlier, we exhibit a collar theorem and a Bers' theorem for cone-surfaces. We describe all cone-surfaces of fixed signature in terms of an underlying combinatorial skeleton and Fenchel-Nielsen parameters. On the spectral side, we give a partial extension of Huber's theorem to Riemann orbisurfaces and use this to show that there are at most finitely many Riemannian orbifolds with the same Laplace spectrum as a given Riemann orbisurface.

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Chapter 1

Introduction

Due in part to their connections with string theory and the study of three-manifolds, the study of singular spaces has been an active area of research in recent years (e.g. [10]). There are still many unanswered questions about the geometry of such spaces; we focus on the case of hyperbolic cone-surfaces with all cone angles less than π. We ask: How do the cone points interact with the closed geodesics on the cone-surfaces? Can we give restrictions on the lengths of certain closed geodesics? Is there a description of all cone-surfaces of fixed signature in terms of a finite set of geometric parameters?

We then specialize to the case of hyperbolic orbifolds. A hyperbolic orbifold can be viewed as the quotient of hyperbolic space by a finite group of isometries, and is thus a visual way to understand a group acting on a space. We will be interested in relationships between the geometric properties of orbisurfaces and the eigenvalue spectrum of the Laplace operator as it acts on smooth functions on the orbisurface. Historically, spectral theory has been concerned with the smooth case. Since Milnor's pair of isospectral (same eigenvalue spectrum) non-isometric flat tori in dimension 16, there has been much work to determine what properties of smooth manifolds are spectrally determined. Recently, the non-smooth case has begun to be studied. For example, Gordon and Rossetti [18] showed that the spectrum of the Hodge Laplacian acting on m-forms on a $2m$-dimensional orbifold cannot distinguish Riemannian manifolds from Riemannian orbifolds. The asymptotic expansion of the heat kernel provides much geometric information in the smooth case; the first few invariants of the heat expansion for orbifolds have been calculated (see [12],[17]). We enter this conversation on the spectral theory of orbifolds by examining the question: Are there geometric restrictions to isospectrality?

The thesis is organized as follows. We begin with a review of concepts from hyperbolic geometry, including a description of several coordinate systems used in the hyperbolic plane, the necessary components of a hyperbolic structure on a given surface, and a discussion of some special types of hyperbolic polygons. We show how to construct the basic building blocks, called "pairs of pants", of our hyperbolic surfaces. The surfaces under consideration have conical singularities; we restrict our attention to those hyperbolic cone-surfaces which are compact, orientable, and have all cone angles less than π. The behavior of geodesics on these so-called "admissible" cone-surfaces is similar to the behavior of geodesics on hyperbolic Riemann surfaces; we modify the relevant theory accordingly.

In Chapter 3, we begin our study of the geometry of admissible cone-surfaces. The collection of pairwise disjoint simple closed geodesics which decompose an admissible cone-surface into pairs of pants is called a *partition*; in joint work with Hugo Parlier, we investigate certain distance sets around these geodesics as well as bounds on the lengths of these geodesics. The collar theorem we prove is a natural generalization of the collar theorem for Riemann surfaces; it shows the existence of disjoint neighborhoods about partitioning geodesics and all cone points on an admissible cone-surface. The neighborhoods about the geodesics are topological cylinders, while those about cone points are cones; the widths found for these neighborhoods are optimal. Bers' theorem, which says that there is a length-bounded partition of a compact Riemann surface, where the length bound depends only on the genus of the surface, has proved to be a useful tool in studying spectral questions on compact Riemann surfaces. In particular, it has been used to find a rough fundamental domain for the action of the Teichmüller modular group, to find an explicit bound on the size of isospectral families, and in estimates involving Fenchel-Nielsen parameters (see [6]). Its utility stems from the fact that it allows one to significantly restrict the allowed lengths of partitioning geodesics. We prove Bers' theorem in the setting of admissible cone-surfaces; our bound depends on the genus and the number of singular points in the cone-surface.

Our study of the geometry of admissible cone-surfaces continues in Chapter 4. We define cubic pseudographs, which serve as the underlying combinatorial skeleton of admissible cone-surfaces. An explicit upper bound on the number of pairwise nonisomorphic cubic pseudographs of fixed signature is exhibited. By gluing pairs of pants along boundary geodesics of the same length according to such a pseudograph, we can form an admissible cone-surface of specified signature. In fact, as we allow the lengths of these geodesics and the twists we introduce in gluing to vary among all possibilities, we obtain all admissible cone-surfaces of a fixed signature.

Beginning with Chapter 5, we restrict our attention to orbifolds, which can be viewed as cone-manifolds in which all cone half-angles are of the form $\frac{\pi}{k}$, for k an integer greater than 1. Hyperbolic orbifolds are spaces which are quotients of hyperbolic space by discrete groups of isometries. We explain how this additional algebraic structure can be exploited to give a convenient generating set for the group of isometries. Finally, we show that every hyperbolic (or Riemann) orbisurface is finitely covered by a smooth Riemann surface.

The study of the spectral theory of Riemann orbisurfaces is the focus of Chapter 6. We begin with the necessary definitions and background concerning the eigenvalue spectrum of the Laplace operator acting on smooth functions on a Riemann orbisurface. Then we move into the study of the relationship between the geometry of an orbisurface and its Laplace spectrum. Using a version of Weyl's asymptotic formula for orbifolds, we are able to give obstructions to isospectrality of Riemann orbisurfaces; these obstructions involve the genus and number of singularities of our orbisurfaces. The length spectrum is the sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order. We show that the Laplace spectrum determines the length spectrum up to finitely many possibilities. Conversely, knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum. In the final section, we use this theorem to show finiteness of isospectral sets of Riemann orbisurfaces.

Chapter 2

Hyperbolic Geometry Background

2.1 The Hyperbolic Plane

Our discussion of this preliminary material closely follows that of Buser [6]. Unless otherwise indicated, we use the Poincaré model of the hyperbolic plane. This model consists of the subset

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

of the complex plane $\mathbb{C}$, together with the metric

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right).$$

Distance in the Poincaré model is given by

$$\cosh \operatorname{dist}(z, w) = 1 + \frac{|z - w|^2}{2\Im z \Im w}.$$

The group

$$PSL(2,\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a,b,c,d \in \mathbb{R}; ad-bc=1\}/\{\pm 1\} = SL(2,\mathbb{R})/\{\pm 1\}$$

acts biholomorphically on $\mathbb{H}$ via the mappings

$$z \mapsto \frac{az+b}{cz+d}.$$

Moreover, $PSL(2,\mathbb{R})$ leaves the hyperbolic metric defined above invariant and is the group of orientation-preserving isometries of $\mathbb{H}$. The geodesics in this model are the generalized circles, i.e. Euclidean circles and lines, which meet the boundary of $\mathbb{H}$ orthogonally. Note that we use $\mathbb{H}$ to denote both the hyperbolic plane and the Poincaré model of the hyperbolic plane; our meaning will be clear from the context.

We recall some basic theorems for the hyperbolic plane.

Theorem 2.1.1. *There exists a unique geodesic between any two distinct points of* $\mathbb{H}$.

Theorem 2.1.2. *Let a be a geodesic and p a point in $\mathbb{H}$. Then there exists a unique geodesic through p perpendicular to a.*

Theorem 2.1.3. *Let a and b be two geodesics in $\mathbb{H}$ with $dist(a,b) > 0$. Then there exists a unique geodesic perpendicular to both a and b.*

2.2 Coordinate Systems

There are several coordinate systems which are frequently used in hyperbolic geometry, including polar coordinates, Fermi coordinates, and horocyclic coordinates (see [6, pp.3-5]). Horocyclic coordinates are primarily useful for the study of surfaces with cusps, which will not be our focus in this work. We now give brief descriptions of polar and Fermi coordinates.

Polar Coordinates

Let $p_0 \in \mathbb{H}$ be fixed, and let v be a unit vector in the tangent space to $\mathbb{H}$ at p_0. For any point $p \in \mathbb{H} \setminus \{p_0\}$, there is a unique geodesic $\eta : [0, \infty) \to \mathbb{H}$ through p with $\eta(0) = p_0$. We assume that η is parametrized by arclength. Let $\sigma \in [-\pi, \pi)$ be the directed angle from v to the tangent vector to η at p_0, and ρ the distance from p_0 to p, i.e. $\eta(\rho(p)) = p$. We call $(\rho, \sigma) = (\rho(p), \sigma(p))$ the polar coordinates of p with respect to p_0 and v. In these coordinates, we write the hyperbolic metric as

$$ds^2 = d\rho^2 + \sinh^2 \rho d\sigma^2. \tag{2.1}$$

Fermi Coordinates

When considering the relation of points to curves in a hyperbolic surface, it is often convenient to use Fermi coordinates. In these coordinates, we replace the base point of polar coordinates by a base line.

Let η be a unit-speed geodesic in the hyperbolic plane. Then η has two "sides", one to the left and one to the right. More precisely, we have the directed distance ρ from a point $p \in \mathbb{H}$ to η; this distance is positive or negative depending on the location of p with respect to η. From the existence of a unique perpendicular from a point to a geodesic in the hyperbolic plane, we know that there is a unique t such that the perpendicular from p to η meets η at time t. We define (ρ, t) to be the pair of Fermi coordinates of p with respect to η. The metric tensor in these coordinates is given by

$$ds^2 = d\rho^2 + \cosh^2 \rho dt^2. \tag{2.2}$$

We will adopt the sign convention that points to the left of an oriented geodesic η will have negative distance ρ to η, while points to the right will have positive ρ-coordinate. This agrees with the convention that points to the left of a counterclockwise-oriented circle (i.e. points inside the circle) have smaller ρ-coordinate than points to the right of, or outside, the circle.

2.3 Hyperbolic Structures

Our goal in this section is to define the hyperbolic structure of two-dimensional hyperbolic cone-manifolds. To do this, we will need to define various types of coordinate charts. We begin by defining the hyperbolic cone and endowing it with a differentiable structure; this will be the key component of a conical chart about a cone point. We follow [11].

Fix $p_0 = i \in \mathbb{H}$. Let $\mu_1, \mu_2 : [0, \infty) \to \mathbb{H}$ be two geodesics such that $\mu_i(0) = p_0$ and γ is the angle between μ_i and the imaginary axis for $i = 1, 2$. We assume that μ_1 and μ_2 are parametrized with respect to arclength and that γ is measured from the imaginary axis. Note that μ_1 and μ_2 emanate from p_0 and are symmetric with respect to the imaginary axis. The isometry

$$z \mapsto m(z) = \frac{(\cos\gamma)z + \sin\gamma}{-(\sin\gamma)z + \cos\gamma}$$

carries μ_1 onto μ_2. The domain

$$\mathcal{D} = \{(\rho, \sigma) \in \mathbb{H} \mid 0 < \rho < \infty,\ -\gamma \leq \sigma \leq \gamma\} \cup \{p_0\}$$

with metric

$$ds^2 = d\rho^2 + \sinh^2 \rho d\sigma^2$$

is a hyperbolic surface. We can glue the boundary of $\mathcal{D}$ under the pasting condition $\mathcal{P}_{\mathcal{D}}$ given by

$$\mu_1(t) = \mu_2(t), \ \forall t \in [0, \infty)$$

and obtain the quotient surface $\mathcal{C}_{p_0} := \mathcal{D} \bmod \mathcal{P}_{\mathcal{D}} = \{[z] | z \in \mathcal{D}\}$ with

(i) $[z] = \{z\}$ if $z \in \text{Int } \mathcal{D}$

(ii) $[z] = \{z, z'\}$ if $z \in \mu_1, \ z' = m(z) \in \mu_2$

(iii) $[p_0] = \{p_0\}$.

We call $\mathcal{C}_{p_0}$ a *conic hyperbolic surface with point* p_0. We can modify our domain $\mathcal{D}$ slightly as follows:

$$\mathcal{D}_\delta := \{(\rho, \sigma) \in \mathbb{H} \,|\, 0 < \rho < \delta, -\gamma \leq \sigma \leq \gamma\} \cup \{p_o\}, \ \delta > 0.$$

Then $\mathcal{D}_\delta \bmod \mathcal{P}_{\mathcal{D}}$, where $t \in [0, \delta)$, is a hyperbolic cone with point p_0 and radius δ, denoted $\mathcal{C}_{p_0,\delta}$.

Next we want to endow $\mathcal{C}_{p_0,\delta}$ with a differentiable structure. Let

$$\bar{\mathcal{D}} = \{(\bar{\rho}, \bar{\sigma}) \in \mathbb{H} \,|\, 0 < \bar{\rho} < \delta, -\pi \leq \bar{\sigma} \leq \pi\} \cup \{p_0\},$$

and define a change of parameter $\bar{\psi} : \mathcal{D}_\delta \to \bar{\mathcal{D}}$ by $\bar{\rho} = \rho, \ \bar{\sigma} = \frac{\pi}{\gamma}\sigma, \ \bar{\psi}(p_0) = p_0$. Then, viewing $\mathcal{C}_{p_0,\delta}$ as a set of equivalence classes of points in $\mathcal{D}_\delta$, we can define a bijection $\psi : \mathcal{C}_{p_0,\delta} \setminus [p_0] \to \bar{\mathcal{D}} \setminus \{p_0\}$ by

$$\psi([z]) = \begin{cases} \bar{\psi}(z) \text{ if } z \in \text{Int } \mathcal{D}_\delta \\ \bar{\psi}(z') \text{ with } z' \in [z] \text{ if } z' \in \mu_1. \end{cases}$$

Since $\bar{\mathcal{D}} \setminus \{p_0\}$ is a differentiable manifold, there is a unique differentiable structure on $\mathcal{C}_{p_0,\delta} \setminus [p_0]$ which makes ψ a diffeomorphism. Our hyperbolic cone $\mathcal{C}_{p_0,\delta}$ with this uniquely defined structure will be used below to construct an atlas on a hyperbolic cone-surface.

For a thorough treatment of hyperbolic structures on Riemann surfaces (with or without boundary), we refer the reader to [6, Ch.1,§2]. Charts on such surfaces will be referred to as *usual charts*. We will now focus on the case of two-dimensional hyperbolic cone-manifolds. Let S be a surface and $p_1, \ldots, p_n$ a collection of cone points on S with $p_i \in \text{Int } S$ for $i = 1, \ldots, n$.

Definition 2.3.1. *Let (U_{i_0}, ϕ_{i_0}) be a pair such that*

(i) $p_{i_0} \in U_{i_0}$ *and* $p_i \notin U_{i_0}$ $\forall i \neq i_0,\ i = 1, \ldots, n$

(ii) $U_{i_0} \subset \text{Int } \ S$

(iii) $\phi_{i_o} : U_{i_0} \to \mathcal{C}_{p_{i_0},\delta}(\gamma_{i_0})$ *is a homeomorphism,* $\phi_{i_0}(p_{i_0}) = p_{i_0}$*, and* $2\gamma_{i_0}$ *is the angle at the point* p_{i_0}*.*

We call (U_{i_0}, ϕ_{i_0}) *a* conical chart *about the point* p_{i_0}*.*

Definition 2.3.2. *Given a surface S with cone points $p_1, \ldots, p_n$, an atlas $\mathcal{A}_{1,2,\ldots,n}$ on S is called* conical hyperbolic *if:*

(i) For every $s \in S, s \neq p_i$, there exists a usual chart $(U, \phi) \in \mathcal{A}_{1,2,\ldots,n}$ with $s \in U$ and $p_i \notin U$ for every $i = 1, \ldots, n$.

(ii) For every $i = 1, \ldots, n$ there exists a conical chart (U_i, ϕ_i) with $p_i \in U_i$.

(iii) Given two charts (U, ϕ) and (U', ϕ') with $U \cap U' \neq \emptyset$, the map $\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$ is an isometry for every connected component $V \subset U \cap U'$.

Definition 2.3.3. *Given a surface S with cone points $p_1, \ldots, p_n$, a conical hyperbolic atlas $\mathcal{A}_{1,2,\ldots,n}$ is called a* conical hyperbolic structure *if it is maximal with respect to the conditions of Definition 2.3.2.*

Such a structure induces on S a metric, which is Riemannian of constant curvature -1 on $S \setminus \{p_1, \ldots, p_n\}$. We say that the structure is *complete* if S is complete in the induced metric.

Definition 2.3.4. *We call a surface S with cone points $p_1, \ldots, p_n$ a* hyperbolic cone-surface *if it is connected and has a complete conical hyperbolic structure.*

The *genus* of a hyperbolic cone-surface is defined as for surfaces, i.e. as the number of "handles" in the cone-surface.

Definition 2.3.5. *Let S be a hyperbolic cone-surface of genus g with m boundary components and n cone points, where each boundary component is a smooth closed geodesic. We say that S is of* signature (g, m, n).

Note that when S is assumed to have empty boundary, we often omit the m from the signature and write the signature of S as (g, n). We define a lexicographic ordering on the triples (g,m,n) as follows:

$$(g, m, n) < (g', m', n') \Leftrightarrow \begin{cases} g < g' \\ \text{or } g = g' \text{ and } \begin{cases} m < m' \\ \text{or } m = m' \text{ and } n < n'. \end{cases} \end{cases}$$

We can also define the area of a hyperbolic cone-surface (see [10]).

Definition 2.3.6. *Let S be a hyperbolic cone-surface of signature (g, n) with cone* *angles* $2\varphi_1, \ldots, 2\varphi_n$. *We define the* area *of* S *as*

$$Area\ (S) = 2\pi(2g-2) + \sum_{i=1}^{n}(2\pi - 2\varphi_i).$$

We will sometimes restrict our attention to cone-surfaces whose cone angles have a special form.

Definition 2.3.7. *Let* S *be a hyperbolic cone-surface with cone points* $p_1, \ldots, p_n$. *If* $\gamma_i = \frac{\pi}{k_i},\ k_i \in \mathbb{N}_+ - \{1\},\ i = 1, \ldots, n$, *where* γ_i *is the half-angle at the point* p_i, *then* S *is called a* Riemann orbisurface.

Note that S is an orbifold with n cone points of orders $k_1, \ldots, k_n$, respectively.

2.4 Building Blocks

In this section we introduce the ideas needed to understand pairs of pants, which are the basic building blocks of hyperbolic cone-surfaces. We begin with an overview of geodesic polygons, followed by a description of how to glue such polygons.

Geodesic Polygons

We now consider hyperbolic geodesic polygons, which are compact domains $P \subset \mathbb{H}$ with closed, piecewise geodesic boundary. We will assume that P is oriented for the following discussion.

In order to understand the decomposition of hyperbolic cone-surfaces into pairs of pants, we must study three special types of geodesic polygons. These are the trirectangle, the generalized quadrorthogonal triangle (GQT) and the geodesic hexagon. We begin with some general terminology.

Let s, t be two consecutive sides of a geodesic polygon P with common endpoint p. We order s and t according to the orientation of P, and denote this ordering by (s,t).

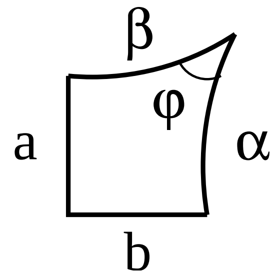

Figure 2.1: Trirectangle

The angle u between s and t is defined to be the angle of the orientation-preserving rotation which carries t to s and fixes p. We say that u is the *subsequent angle* of side s and t is the *subsequent side* of angle u.

Definition 2.4.1. *Let x and y be among the set of sides and angles of P. We say that the ordered pair (x,y) is* of angle type *if (x,y) satisfies one of the following conditions:*

1. *The angle y is the subsequent angle of the side x.*

2. *(x,y) is a pair of consecutive sides, and x and y are orthogonal.*

3. *The side y is the subsequent side of the angle x.*

Let P be a geodesic polygon described by the cycle $a\gamma b\alpha c\beta$, where each element in the list is either a side or an angle of P. We say that P is a *generalized triangle* if every pair of consecutive elements in this cycle is of angle type.

Definition 2.4.2. *Let P be a geodesic polygon given by $ab\alpha\varphi\beta$. If there is exactly one pair in this list of angle type (1), we say that P is a* trirectangle. *Note that P has three right angles and four sides (see Figure 2.1).*

We usually let φ denote the non-right angle and a, b, α, β the sides. Note that we use this notation to mean both the label of the side (angle) and its length (measure). Buser [6, p. 38] gives the following relationships among these various quantities:

Theorem 2.4.3. *For every trirectangle with sides labelled as in Figure 2.1, we have*

1. $\cos\varphi = \sinh a \sinh b$
2. $\cos\varphi = \tanh\alpha \tanh\beta$
3. $\cosh a = \cosh\alpha \sin\varphi$
4. $\cosh a = \tanh\beta \coth b$
5. $\sinh\alpha = \sinh a \cosh\beta$
6. $\sinh\alpha = \coth b \cot\varphi$.

Note that we can specify two of the five parameters which define a trirectangle, and that this choice uniquely determines the remaining parameters. In particular, we can specify φ and a. From Theorem 2.4.3 (i) we see that

$$b = \operatorname{arcsinh}\left(\frac{\cos\varphi}{\sinh a}\right),$$

and for fixed φ, as a ranges over $\mathbb{R}_+$, so does b. We now turn to generalized quadrorthogonal triangles (GQTs).

Definition 2.4.4. *Let P be a geodesic polygon given by $a\gamma b\alpha c\beta$. If there is exactly one pair in this list of angle type (1), we say that P is a* generalized quadrorthogonal triangle (GQT).

Note that P has four right angles and five sides; that is, P is a geodesic pentagon. See Figure 2.2.

We usually let γ denote the non-right angle and a, b, α, c, β the sides. These quantities are related as follows ([6, p. 37]).

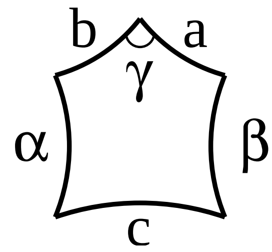

Figure 2.2: Generalized Quadrorthogonal Triangle

Theorem 2.4.5. *For every GQT $a\gamma b\alpha c\beta$ as in Figure 2.2, the following relationships hold:*

1. $\cosh c = -\cosh a \cosh b \cos\gamma + \sinh a \sinh b$

2. $\cosh a : \sinh\alpha = \cosh b : \sinh\beta = \sinh c : \sin\gamma$

3. $\cos\gamma = \sinh\alpha \sinh\beta \cosh c - \cosh\alpha \cosh\beta$

Dianu showed the following theorem regarding the existence of GQT's ([11, p.7]):

Theorem 2.4.6. *Let $x, y > 0$ and $z \in (0, \gamma_0)$ with $\cos\gamma_0 = -\frac{\cosh min(x,y)}{\cosh max(x,y)}$. Then there exists a unique GQT with $a\gamma b\alpha c\beta$ such that $\alpha = x$, $\beta = y$, and $\gamma = z$.*

Note that since $\cosh u \geq 1\ \forall u \in \mathbb{R}$, we have $\cos\gamma_0 < 0$. Thus $\gamma_0 > \frac{\pi}{2}$, and there is a unique GQT for any specified $x, y > 0$ and any cone half-angle in $(0, \frac{\pi}{2})$. As noted in [11], if $\beta < \alpha$ then for $\gamma = \gamma_0$, our GQT becomes a trirectangle with acute angle $\theta = \gamma_0 - \frac{\pi}{2}$. For γ such that $\cos\gamma \in (-1, \cos\gamma_0]$, our GQT is self-intersecting.

We next consider geodesic hexagons, where we use hexagon to refer to both the domain and its boundary. We say that a geodesic hexagon P is *convex* if it is convex as a domain. In geodesic hexagons, every pair in the cycle $a\gamma b\alpha c\beta$ is of angle type (2). See Figure 2.3.

We have relationships among the lengths of the sides of a geodesic hexagon ([6, p.40]):

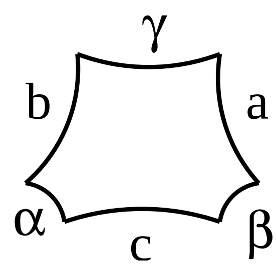

Figure 2.3: Hexagon

Theorem 2.4.7. *Let P be a convex right-angled geodesic hexagon with consecutive sides $a\gamma b\alpha c\beta$as in Figure 2.3. Then the following hold:*

1. $\cosh c = \sinh a \sinh b \cosh\gamma - \cosh a \cosh b$

2. $\sinh a : \sinh\alpha = \sinh b : \sinh\beta = \sinh c : \sinh\gamma$

3. $\coth\alpha \sinh\gamma = \cosh\gamma \cosh b - \coth a \sinh b$

Additionally, we can make the following statement regarding the existence of convex right-angled geodesic hexagons (see [6, p.40]):

Theorem 2.4.8. *Let $x, y, z > 0$ be real numbers. Then there exists a unique convex right-angled geodesic hexagon $a\gamma b\alpha c\beta$ with $x = a$, $y = b$, $z = c$.*

Pasting

We now consider the process of pasting geodesic polygons along sides of equal length (cf. [6]). Let P and P' be two disjoint convex geodesic polygons with sides γ and γ' of length l. Suppose that $\gamma : [0,1] \to P$ and $\gamma' : [0,1] \to P'$ are parametrized with constant speed and opposite orientation. Then there is an isometry $m \in PSL(2,\mathbb{R})$ which carries γ to γ', i.e. $m(\gamma(t)) = \gamma'(t)$ for all $t \in [0,1]$. We can define an equivalence relation on $P \cup P'$ as follows:

$$[p] = \begin{cases} \{p\} & \text{if } p \in P \cup P' - \{\gamma \cup \gamma'\}; \\ \{p, p'\} & \text{if } \exists t_0 \in [0,1] \text{ such that } p = \gamma(t_0),\ p' = \gamma'(t_0). \end{cases}$$

The above equivalence relation is said to be determined by the pasting condition $\mathcal{P}_{P+P'}$ given by

$$\gamma(t) = \gamma'(t),\ t \in [0,1].$$

Note that $F := P + P' \bmod \mathcal{P}_{P+P'}$ is isometric to $m(P) \cup P'$.

We can, in a similar fashion, paste together a collection of hyperbolic surfaces. Let $S_1, S_2, \ldots, S_m$ be a set of disjoint hyperbolic surfaces with pairwise disjoint piecewise smooth sides $\gamma_1, \gamma_1', \gamma_2, \gamma_2', \ldots, \gamma_n, \gamma_n'$. We suppose that γ_k, γ_k' are parametrized with the same constant speed on an interval I_k, i.e. $\gamma_k : I_k \to S$, where $S := S_1 \cup S_2 \cup \cdots \cup S_m$. When γ_k and γ_k' are smooth closed geodesics, we allow I_k to be the interval $(-\infty, +\infty)$ and we parametrize the geodesics periodically. The orientation of the geodesics is assumed to be such that the quotient surface defined below is orientable. We define the pasting condition $\mathcal{P}_S$ on S to be

$$\gamma_k(t) = \gamma_k'(t),\ t \in I_k,\ k = 1, \ldots, n.$$

As above, this condition determines an equivalence relation on S, and we have the quotient space $F := S_1 + S_2 + \cdots + S_m \bmod \mathcal{P}_S$. In the following theorem (see [6, p.13]) we see that F is a hyperbolic surface. Note that a *vertex cycle* is the set of all vertices of $S_1, S_2, \ldots, S_m$ which define a single point of F (where the point can lie either in the interior or on the boundary of F).

Theorem 2.4.9. *Suppose that the following conditions hold:*

(i) For every vertex cycle which gives rise to an interior point of F, the sum of the interior angles at the vertices is equal to 2π.

(ii) For every vertex cycle which gives rise to a boundary point of F, the sum of the interior angles at the vertices is less than or equal to π.

Then there is a unique hyperbolic structure on F such that the projection $\sigma : S_1 \cup S_2 \cup \cdots \cup S_m \to F$ is a local isometry. Suppose that, in addition, we have:

(i) F is connected.

(ii) The hyperbolic structure on S_k is complete for $k = 1, \ldots, m$.

(iii) For any pair of non-adjacent sides in the list $\gamma_1, \gamma_1', \ldots, \gamma_n, \gamma_n'$ which lie on the same S_k, there is positive distance between the sides.

Then the hyperbolic structure on F is complete.

To conclude our discussion of pasting, we define the inverse process of cutting.

Definition 2.4.10. *Let F be as above, and $\sigma : S_1 \cup S_2 \cup \cdots \cup S_m \to F$ the natural projection. Define*

$$C = \sigma(\gamma_1) \cup \sigma(\gamma_2) \cup \cdots \cup \sigma(\gamma_n) = \sigma(\gamma_1') \cup \sigma(\gamma_2') \cup \cdots \cup \sigma(\gamma_n').$$

Then we can recover $S_1, S_2, \ldots, S_m$ by cutting F open along C.

2.5 Pairs of Pants

The basic building blocks for admissible cone-surfaces are Y-pieces, V-pieces, and joker's hats, collectively called "pairs of pants." With our understanding of geodesic polygons, pasting, and hyperbolic structures, we can now construct these pieces.

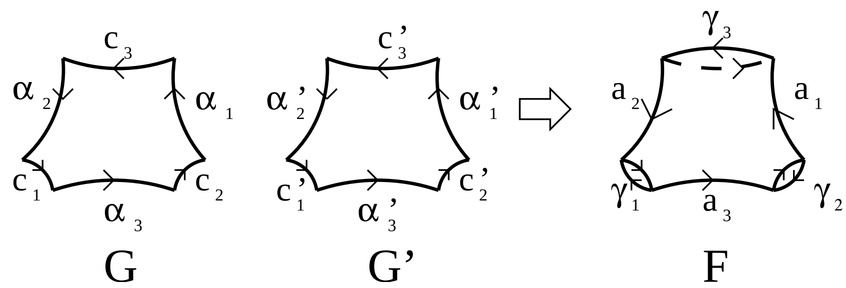

Figure 2.4: Gluing Hexagons

Let G and G' be two copies of a right-angled geodesic hexagon in the hyperbolic plane. The sides of G (respectively, G') are labelled by the cycle $\alpha_1 c_3 \alpha_2 c_1 \alpha_3 c_2$ (respectively, $\alpha'_1 c'_3 \alpha'_2 c'_1 \alpha'_3 c'_2$). See Figure 2.4. We assume that all sides are parametrized on the interval [0,1] with constant speed. The pasting condition $\mathcal{P}_{G+G'}$ is

$$\alpha_i(t) = \alpha'_i(t) := a_i(t),\ i = 1, 2, 3,\ t \in [0, 1].$$

The quotient space $F := G + G' \bmod \mathcal{P}_{G+G'}$ is a hyperbolic surface with three smooth closed boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ defined by

$$t \mapsto \gamma_i(t) := \begin{cases} c_i(2t) & ,\ 0 \leq t \leq \frac{1}{2} \\ c'_i(2 - 2t) & ,\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $i = 1, 2, 3$.

Definition 2.5.1. *A hyperbolic surface of type (0,3,0) is called a* Y-piece.

Every Y-piece is obtained by gluing two isometric right-angled geodesic hexagons as above (see [6, §3.1]). We can also decompose every Y-piece into such hexagons in a canonical fashion (see [6, Prop. 3.1.5]):

Proposition 2.5.2. *Let S be a given Y-piece. For every pair of boundary geodesics of S there exists a unique simple common perpendicular. The three perpendiculars together decompose S into two isometric right-angled geodesic hexagons.*

We have the following statement regarding the possible lengths for the boundary components of a pair of pants:

Theorem 2.5.3. *Fix $l_1, l_2, l_3 \in \mathbb{R}_+$. Then there exists a unique Y-piece with boundary geodesics of lengths l_1, l_2, l_3.*

The construction of V-pieces is similar; we now use GQT's as our geodesic polygons. Let T be a GQT described by the cycle $\mu_1\gamma\mu_2\eta_1\mu_3\eta_2$, and T' a copy of T with cycle $\mu_1'\gamma\mu_2'\eta_1'\mu_3'\eta_2'$. See Figure 2.5. We assume that all sides are parametrized on the unit interval with constant speed. The pasting conditions are:

$$\mu_i(t) = \mu_i'(t) := \begin{cases} a(t) & \text{if } i = 1 \\ b(t) & \text{if } i = 2 \\ c(t) & \text{if } i = 3 \end{cases} \qquad \forall t \in [0,1]$$

and

$$t \mapsto \alpha(t) := \begin{cases} \eta_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \eta_1'(2-2t) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$t \mapsto \beta(t) := \begin{cases} \eta_2(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \eta_2'(2-2t) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The quotient surface F thus obtained has a cone point with angle 2γ and two boundary components which are smooth closed geodesics.

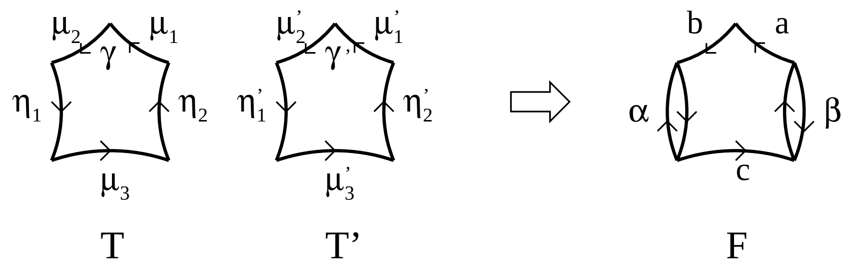

Figure 2.5: Gluing GQT's

Definition 2.5.4. *An admissible cone-surface S of signature $(0,2,1)$ is called a V-piece.*

As was the case with Y-pieces, every V-piece is obtained by gluing two isometric GQT's in this way. There is also a canonical decomposition of a given V-piece into two such GQT's (see [11]).

Proposition 2.5.5. *Let S be a given V-piece. There exists a unique simple common perpendicular between the two boundary geodesics of S. There exist unique simple geodesics from the cone point of S to the boundary geodesics, each of which is perpendicular to its respective boundary geodesic. The three perpendiculars together decompose S into two isometric GQT's.*

From Theorem 2.4.6, we see that

Theorem 2.5.6. *Fix $\ell_1, \ell_2 \in \mathbb{R}_+$ and $\gamma \in (0, \gamma_0)$ with $\cos\gamma_0 = -\frac{\cosh min(\ell_1/2, \ell_2/2)}{\cosh max(\ell_1/2, \ell_2/2)}$. Then there exists a unique V-piece with boundary geodesics of lengths ℓ_1, ℓ_2 and cone angle 2γ.*

Finally, we construct joker's hats, using trirectangles as our basic geodesic polygons. Let R_1 be a trirectangle described by the cycle $a_1 b_1 \alpha_1 \varphi_1 \beta_1$, and let R_2 be a trirectangle described by the cycle $a_2 b_2 \alpha_2 \varphi_2 \beta_2$. If $a_1 = a_2$, then we can glue R_1 to R_2

along a_1 and a_2, which we assume are parametrized on the unit interval with constant speed. We define the pasting condition $\mathcal{P}_{R_1+R_2}$ by

$$a_1(t) = a_2(t) := a(t), \quad t \in [0,1]. \tag{2.3}$$

Define $R := R_1 + R_2 \ mod(2.3)$,and let $b(t) \in R$ be the geodesic obtained by first traversing $b_1(t)$ and then traversing $b_2(t)$, and similarly for $\beta(t) \in R$. We assume that all sides of R are parametrized on $[0,1]$ with unit speed. Let R' be a copy of R. Then we can glue R and R' with pasting conditions

$$\beta(t) = \beta'(t) := p(t) \quad \forall t \in [0,1]$$

$$\alpha_i(t) = \alpha_i'(t) := \begin{cases} q_1(t) & \text{if } i = 1 \\ q_2(t) & \text{if } i = 2 \quad \forall t \in [0,1] \end{cases}$$

and

$$t \mapsto r(t) := \begin{cases} b(t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ b'(t) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The quotient surface F thus obtained has two cone points with cone angles $2\varphi_1$ and $2\varphi_2$, and one boundary component which is a smooth closed geodesic.

Definition 2.5.7. *An admissible cone-surface S of signature $(0,1,2)$ is called a* joker's hat.

We can make similar observations as in the cases of Y-pieces and V-pieces: every joker's hat is obtained via this construction, and there is a canonical decomposition of a joker's hat into such pieces. Finally, for any choice of acute angles φ_1 and φ_2, it follows from the remark following Theorem 2.4.3 that the length of $r(t)$ can be chosen to be any positive real number.

2.6 Geodesics on Hyperbolic Cone-Surfaces

Many of our subsequent results require an understanding of the behavior of geodesics on hyperbolic cone-surfaces. We now give the necessary background.

Assumption 2.6.1. *Henceforth, we will assume that all our hyperbolic cone-surfaces are compact and orientable. If S is a hyperbolic cone-surface in which all cone angles are less than π, we will call S an* admissible cone-surface*.*

We begin our exploration of geodesic behavior with a definition.

Definition 2.6.2. *Let M be a topological space, and let $\mu_1, \mu_2 : \mathbb{S}^1 \to M$ be two closed curves on M. We say that μ_1 and μ_2 are* freely homotopic *if there exists a continuous map $c : [0,1] \times \mathbb{S}^1 \to M$ such that*

$$c(0,t) = \mu_1(t) \text{ and } c(1,t) = \mu_2(t) \quad \forall t \in \mathbb{S}^1.$$

Suppose that S is an admissible cone-surface. We denote the set of all cone points on S by Σ. We say that two curves in S are *freely homotopic* if they are freely homotopic on $S \setminus \Sigma$. If a curve in S is freely homotopic to a point in $S \setminus \Sigma$, then it is said to be *homotopically trivial*. A closed curve and a cone point are freely homotopic if they are freely homotopic on S. By curve or geodesic, we mean the set of all points lying on the curve. We have the following properties of closed curves.

Proposition 2.6.3. *Let S be an admissible cone-surface.*

1. *Every non-trivial simple closed curve on $S \setminus \Sigma$ is freely homotopic to either a unique simple closed geodesic or a unique cone point. (For a closed curve δ, the*

associated closed geodesic will be denoted $\mathcal{G}(\delta)$.)

2. *Given two non-intersecting smooth simple curves α and β on S there is at least one geodesic path c between them such that $d_S(\alpha, \beta)$ is realized by c. Such a path c is perpendicular to α and β. If α and β are geodesic, in a free homotopy class of paths with end points moving on α and β, such a path c is unique. This property remains true for singular points in place of one or both geodesics.*

These properties result from the constant negative curvature on an admissible cone-surface S, and can be proved as in the case of compact hyperbolic Riemann surfaces. We prove the first statement to give an indication of the arguments used.

Proof. Let c be a non-trivial simple closed curve on $S \setminus \Sigma$, and let C be the free homotopy class of c. We can find a sequence of smooth closed curves $\{\gamma_n\}_{n=1}^{\infty} \subset C$ such that the lengths of γ_n converge to the infimum L as $n \to \infty$. Assume that the γ_n are parametrized with constant speed on $\mathbb{S}^1$. Then the Arzelà-Ascoli theorem gives the existence of a subsequence which converges to a rectifiable curve $\gamma : \mathbb{S}^1 \to S$ with $\ell(\gamma) = L$. If $L = 0$, then c is homotopic to a cone point. If $L > 0$, then it follows that γ is a geodesic from the minimality of L. The argument that γ is simple is exactly the same as in the proof of Buser's Thm. 1.6.6(iii). $\square$

In addition to knowing how curves on cone-surfaces behave under homotopies, we need to know about the existence of certain curves. Dianu ([11]) showed the following theorem.

Theorem 2.6.4. *Let S be an admissible cone-surface of signature $(g, m, n) > (0, 2, 1)$, $(g, m, n) \neq (0, 3, 0)$. Then there exists a simple closed curve ν on S which is not homotopic to a boundary component or to a cone point of S. Furthermore, ν does not pass through any of the cone points of S.*

This theorem can easily be extended:

Proposition 2.6.5. *Let S be an admissible cone-surface of signature* $(0,0,4) \leq (g,m,n) < (0,1,0)$, $(0,1,2) < (g,m,n) < (0,2,0)$, $(g,m,n) > (0,2,1)$, *where* $(g,m,n) \neq (0,3,0)$. *Then there exists a simple closed curve ν on S which is not homotopic to a boundary component or to a cone point of S. Furthermore, ν does not pass through any of the cone points of S.*

Proof. The only cases to prove are for signature $(0,0,k), k \geq 4$, and $(0,1,l), l \geq 3$. Note that there are always at least two cone points on such a surface; choose two cone points p_1 and p_2. Let δ_i be a circle centered at p_i of radius $\epsilon_i > 0$, $i = 1,2$, and let σ be the common perpendicular of δ_1 and δ_2. Define $\mu = \delta_1 \circ \sigma \circ \delta_2 \circ \sigma^{-1}$, and note that μ is homotopic to a nontrivial simple closed curve ν on $S \setminus \Sigma$. The signature of S implies that ν is not homotopic to a boundary component. $\square$

Henceforth, we make the convention that an admissible cone-surface has one of the following signatures: $(0,0,k), k \geq 4, (0,1,l), l \geq 2$, or $(g,m,n) \geq (0,2,1), (g,m,n) \neq (1,0,0)$.

Chapter 3

Partitions

An admissible cone-surface can be decomposed into pairs of pants. The geodesics involved in such a decomposition have several nice properties; in this chapter, we investigate certain distance sets around these geodesics, and bounds on the lengths of such geodesics. The material in this chapter is joint work with Hugo Parlier (see [13]).

3.1 Decomposition of Admissible Cone-Surfaces

It is well-known that every compact Riemann surface of genus greater than one can be decomposed into Y-pieces (e.g. [6]). Dianu ([11]) showed that an admissible Riemann orbisurface can be decomposed into Y-pieces and V-pieces. Specifically, he proved:

Theorem 3.1.1. *Let S be an admissible Riemann orbisurface of type $(g, m, n) \geq (0, 2, 1)$. Then there exists a decomposition of S into $2g - 2 + m$ Y-pieces and n V-pieces by $3g - 3 + m + n$ simple closed disjoint geodesics.*

We give a sketch of the proof for completeness.

Proof. (Sketch) The argument is by induction on the type (g, m, n) of the surface S. Consider the curve ν of Theorem 2.6.4. Cut S open along ν. The resulting surface S' either remains connected or separates into two connected components. We apply the induction hypothesis in each case, using the additional facts that ν is not homotopic to a boundary component or to a cone point of S in the latter situation. This gives that S can be decomposed into pieces which are topologically equivalent to Y-pieces and V-pieces.

Then there is some work to show that we can replace ν by a geodesic in its homotopy class. Take a sequence of curves in the free homotopy class of ν whose lengths converge to the infimum of their lengths. The Arzela-Ascoli theorem gives the existence of a subsequence converging to some curve $\tilde{\nu}$, and the lower semicontinuity of the length function guarantees that $\tilde{\nu}$ is minimal, with length equal to the infimum.

We then argue that $\tilde{\nu}$ is in the free homotopy class of ν, using a proof by contradiction of the minimality of $\tilde{\nu}$ and the fact that during a free homotopy, curves on S cannot pass over cone points. To show that $\tilde{\nu}$ is simple, we use the universal cover of S (possible because S is an orbifold) and give the standard argument from the case of hyperbolic surfaces (e.g. [6, p.20]).

Finally we invoke Epstein's theorem (see [6]) to get a homeomorphism which fixes the boundary components and cone points of S and takes ν onto $\tilde{\nu}$. Using $\tilde{\nu}$ as our geodesic of decomposition, we see that S can be decomposed into $2g-2+m$ Y-pieces and n V-pieces. □

Dianu's proof of this theorem includes the case of cone points of order 2, but that is not our focus. In fact, Dianu's theorem holds in the more general context of admissible cone-surfaces. The proof only required the hypothesis of orbifold cone angles in showing that $\tilde{\nu}$ is simple. We have the following generalization.

Theorem 3.1.2. *Let S be an admissible cone-surface of signature $(g, m, n) > (0, 2, 1)$. Then there exists a decomposition of S into $2g - 2 + m$ Y-pieces and n V-pieces.*

Proof. The proof is exactly as in the case of an admissible Riemann orbisurface, except that we must show that $\tilde{\nu}$ is simple without resorting directly to the universal cover.

Note that $\tilde{\nu}$ does not pass through any cone points. Thus, we can remove small neighborhoods of all the cone points without interfering with our decomposing geodesic. The surface that results is a hyperbolic surface with smooth (although not geodesic) boundary. We can then invoke a theorem of Buser ([6, Thm. 1.6.6]) to conclude that $\tilde{\nu}$ is simple. Epstein's theorem can be applied as above to get a decomposition of S into $2g - 2$ Y-pieces and n V-pieces. $\square$

Thurston [33] noted that there exists a decomposition of an admissible Riemann orbisurface of any permitted signature (see the convention at the end of Chapter 2) into pairs of pants, including joker's hats. This was based on a decomposition of the orbifolds into primitive pieces; we can carry out the same decomposition for admissible cone-surfaces. The only modification required in the proof of Theorem 3.1.2 is in the first step, that of finding an initial simple closed geodesic along which to cut. By Proposition 2.6.5, an admissible cone-surface S always contains a simple closed curve which is not homotopic to a boundary component or to a cone point; we take this as our initial geodesic. So an admissible cone-surface can be decomposed into pairs of pants. A collection of pairwise disjoint simple closed geodesics which decompose a (cone-)surface into pairs of pants is called a *partition*.

3.2 Collars

The collar theorem for compact hyperbolic Riemann surfaces (e.g. [21], [5]) states that surrounding a simple closed geodesic there is a tubular neighborhood, called a collar, which is a topological cylinder. This neighborhood is of a certain width which depends uniquely on the length of the geodesic. Furthermore, if two simple closed geodesics do not intersect then their collars are disjoint. Finally, the values given for the widths of the collars are optimal (e.g. [31], [26]). There has been interest in proving a similar theorem for orbifolds (e.g. [11], [16] and [23]), where the object was often to estimate minimal distance between singular points based on the order of the points. The collar theorem for admissible cone-surfaces has the same properties as the original collar theorem, and is thus a natural generalization.

Theorem 3.2.1. *Let S be an admissible cone-surface of signature (g, n) with cone points $p_1, \ldots, p_n$ and cone angles $2\varphi_1, \ldots, 2\varphi_n$. Let 2φ be the largest cone angle. Let $\gamma_1, \ldots, \gamma_m$ be disjoint simple closed geodesics on S. Then the following hold.*

1. *$m \leq 3g - 3 + n$.*
2. *There exist simple closed geodesics $\gamma_{m+1}, \ldots, \gamma_{3g-3+n}$ which together with $\gamma_1, \ldots, \gamma_m$ form a partition of S.*
3. *The collars*

$$\mathcal{C}(\gamma_k) = \{x \in S \mid d(x, \gamma_k) \leq w_k = \operatorname{arcsinh}(\cos\varphi / \sinh\frac{\gamma_k}{2})\}$$

and

$$\mathcal{C}(p_l) = \{x \in S \mid d(x, p_l) \leq v_l = \operatorname{arccosh}(1/\sin\varphi_l)\}$$

are pairwise disjoint for $k = 1, \ldots, 3g - 3 + n$ and $l = 1, \ldots, n$.

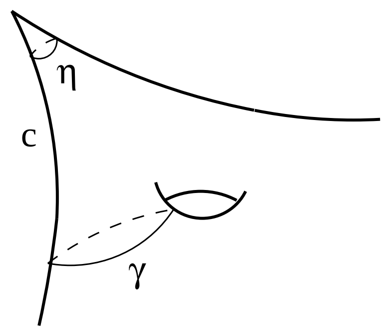

Figure 3.1: Determining a unique pair of pants

4. Each $\mathcal{C}(\gamma_k)$ *is isometric to the cylinder* $[-w_k, w_k] \times \mathbb{S}^1$ *with the Riemannian metric* $ds^2 = d\rho^2 + \ell^2(\gamma_k)\cosh^2\rho dt^2$*.*

Each $\mathcal{C}(p_l)$ *is isometric to a hyperbolic cone* $[0, w_l] \times \mathbb{S}^1$ *with the Riemannian metric* $ds^2 = d\rho^2 + \frac{\varphi_l^2}{\pi^2}\sinh^2\rho dt^2$*.*

Before proceeding to the proof, we must introduce the following construction. Let p be a cone point and γ a simple closed geodesic on S. Let c be a simple geodesic path from p to γ, perpendicular to γ. These elements describe a unique pair of pants in the following manner. Let δ be the closed curve obtained by taking $\gamma \circ c \circ \eta \circ c^{-1}$ as in Figure 3.1.

Then $\mathcal{G}(\delta)$ is either a simple closed geodesic or another cone point, and in either case $(p, \gamma, \mathcal{G}(\delta))$ is a pair of pants. Using exactly the same technique with two cone points and a simple geodesic path between them, or with two simple closed geodesics and a perpendicular simple geodesic path between them, we get pairs of pants that are uniquely determined. We will use this construction in the following proof of the collar theorem for admissible cone-surfaces.

Proof. The first two points are equivalent to the problem of counting the number of geodesics in a partition for a surface of signature (g, n), and showing that any

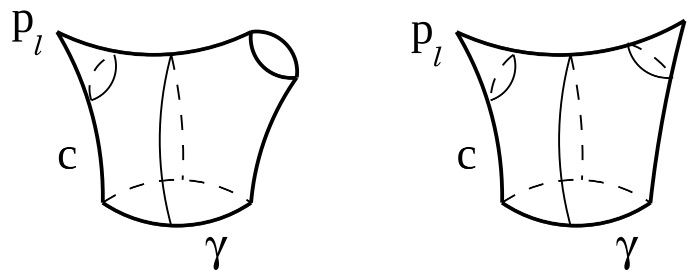

Figure 3.2: V-piece and Joker's Hat

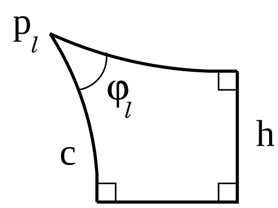

Figure 3.3: Extracted Trirectangle

collection of pairwise disjoint simple closed geodesics can be completed to form a partition. These questions are not new and the proofs are known (e.g. [6, p.112]).

The first step in proving the theorem is to show that a cone point p_l is at a distance of at least $\operatorname{arccosh}(1/\sin\varphi_l)$ from all simple closed geodesics. Let γ be a simple closed geodesic on S. Take a geodesic path c that realizes the distance between p_l and γ (i.e. such that $\ell(c) = d(p_l, \gamma)$). Then take the unique pair of pants obtained from p_l, γ and $\mathcal{G}(\gamma \circ c \circ \eta \circ c^{-1})$ as discussed previously. Either this pair of pants is a V-piece or a joker's hat. In both cases extract the trirectangle as in Figures 3.2 and 3.3.

From Theorem 2.4.3 (3) we have

$$\cosh c = \frac{\cosh h}{\sin\varphi_l}$$

where h is as in figure 3.3. It follows that $c > \operatorname{arccosh}(1/\sin\varphi_l)$.

Let p_l and $p_{l'}$ be two distinct cone points. Let b be a path that realizes distance

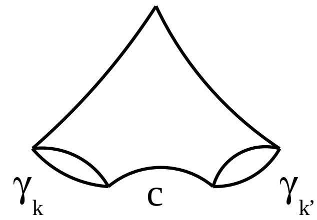

Figure 3.4: V-piece

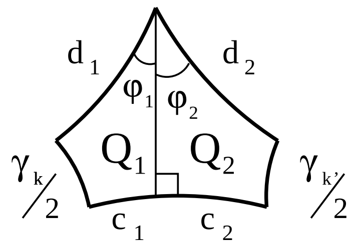

Figure 3.5: Extracted GQT

between them. Let γ be a simple closed geodesic that crosses b; such a geodesic must exist by Proposition 2.6.5. It is clear that $\ell(b)$ is necessarily greater than or equal to the distance from p_l to γ added to the distance from $p_{l'}$ to γ. From what precedes we have:

$$\ell(b) \geq d(p_l, \gamma) + d(p_{l'}, \gamma) > \operatorname{arccosh}\left(\frac{1}{\sin \varphi_l}\right) + \operatorname{arccosh}\left(\frac{1}{\sin \varphi_{l'}}\right).$$

It follows that the distance sets $\mathcal{C}(p_l)$ and $\mathcal{C}(p_{l'})$ are disjoint.

Let γ_k and $\gamma_{k'}$ be two disjoint simple closed geodesics. Let c be a path that realizes distance between them. The unique pair of pants determined by c, γ_k and $\gamma_{k'}$ is either a Y-piece or a V-piece. In the first case it follows that $\mathcal{C}(\gamma_k)$ and $\mathcal{C}(\gamma_{k'})$ are disjoint sets from the collar theorem on Riemann surfaces. In the latter case, consider Figures 3.4 and 3.5.

Notice that only half of the collar around a given boundary geodesic is contained in a pair of pants. Also note that both angles φ_1 and φ_2 are strictly inferior to φ.

From Theorem 2.4.3 (1) applied to Q_1 we obtain

$$\sinh c_1 = \frac{\cos \varphi_1}{\sinh \frac{\gamma_k}{2}},$$

and analogously for Q_2. From this we obtain that

$$c_1 = \operatorname{arcsinh} \left(\frac{\cos \varphi_1}{\sinh \frac{\gamma_k}{2}} \right),$$

and

$$c_2 = \operatorname{arcsinh} \left(\frac{\cos \varphi_2}{\sinh \frac{\gamma_{k'}}{2}} \right).$$

It follows that the distance sets $\mathcal{C}(\gamma_k)$ and $\mathcal{C}(\gamma_{k'})$ are disjoint, because each collar is the union of two such half-collars.

It remains to prove that for arbitrary p_l and γ_k the collars are disjoint. Let c be a geodesic path that realizes the distance between them. The collars around both the cone point and the geodesic have widths which depend only on their angle or length. It thus suffices to consider the case where the cone point and the geodesic would be as close as possible. This case would occur if the pair of pants resulting from p_l, γ_k and c were a joker's hat with both cone angles equal to 2φ. To see this, consider Figure 3.3. Let h grow continuously; as it approaches the limiting case (when φ_l reaches the endpoint at infinity of c), φ_l tends to zero. Conversely, as φ_l grows, h and c shrink, bringing the cone point closer to the relevant geodesic. In the case of a joker's hat with both cone angles equal to 2φ, Figure 3.6 would apply.

From Theorem 2.4.3 (6) one obtains

$$\sinh c = \coth \frac{\gamma_k}{4} \cot \varphi.$$

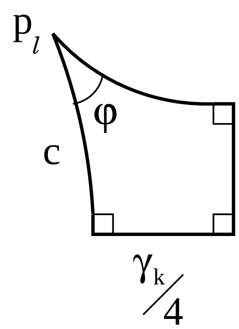

Figure 3.6: Trirectangle with acute angle φ

This in turn can be expressed as

$$\sinh c = \frac{\cosh \frac{\gamma_k}{2} + 1}{\sinh \frac{\gamma_k}{2}} \cot \varphi.$$

Let us compare this with the value obtained by calculating $\sinh(w_k + v_l)$. By calculation one obtains

$$\sinh(w_k + v_l) = \frac{1 + \sqrt{\cos^2 \varphi + \sinh^2 \frac{\gamma_k}{2}}}{\sinh \frac{\gamma_k}{2}} \cdot \cot \varphi.$$

By comparison and because $\varphi < \frac{\pi}{2}$ we obtain that $c > w_k + v_l$. This implies that the distance sets $\mathcal{C}(\gamma_k)$ and $\mathcal{C}(p_l)$ are disjoint.

The last point of the theorem is obtained as in the classical theory of Riemann surfaces for $\mathcal{C}(\gamma_k)$ (see [6]), and was shown by Dianu ([11]) for $\mathcal{C}(p_l)$. $\square$

An example of the utility of Theorem 3.2.1 is the following natural corollary.

Corollary 3.2.2. *Let γ and δ be closed geodesics on S which intersect each other transversally, and assume that γ is simple. Then*

$$\sinh \frac{\ell(\gamma)}{2} \sinh \frac{\ell(\delta)}{2} > \cos \varphi.$$

Remark 3.2.3. *The values for the collars are optimal in the following sense. The collar around a simple closed geodesic γ can be seen as a distance set with the following property: if another simple closed geodesic enters $\mathcal{C}(\gamma)$, then it necessarily intersects γ. Replacing $w_\gamma = \operatorname{arcsinh}(\cos\varphi/\sinh\frac{\gamma}{2})$ by $w = w_\gamma + \varepsilon$ with $\varepsilon > 0$ would be fatal to this property. In fact, for any $\varepsilon > 0$ there are an infinity of simple closed geodesics that intersect the enlarged collar but not γ. To prove this, take a V-piece containing γ and p where the cone angle at p is exactly 2φ. The other boundary geodesic γ' of the V-piece can be chosen as long as desired (see [25]). From the formulas obtained in the proof, it follows that γ' can be arbitrarily close to the collar of γ. What is striking is that the sharp bound for the width of the collar is independent of the location of the geodesic on the surface. In an analogous fashion, one can show that the bound for collars around cone points is also sharp.*

3.3 Bers' Theorem

Bers' theorem states that there is a length-bounded partition of every compact Riemann surface of genus $g \geq 2$, where the length bound is a constant depending only on g. Recall that a partition is a collection of pairwise disjoint simple closed geodesics which decompose the surface into pairs of pants; in the case of Riemann surfaces, a partition is a decomposition into Y-pieces. Buser gave an explicit bound for the lengths of the geodesics in such a partition, and also gave a bound for the lengths of decomposing geodesics in the non-compact case (see [6, Ch. 5]). In this section we prove an analogous result in the setting of admissible cone-surfaces. Specifically, we show

Theorem 3.3.1. *Let S be a compact admissible cone-surface of signature (g, n). Then there exists a partition $\mathcal{P}$ of S such that every geodesic in $\mathcal{P}$ has length less than a* *constant* $L_{g,n}$.

Proof. Let $p_1, \ldots, p_n$ be the cone points on S with corresponding cone angles $2\varphi_1, \ldots, 2\varphi_n$. We define

$$Z_i(r_i) = \{x \in S | \operatorname{dist}(x, p_i) \leq r_i\},$$

for $i = 1, \ldots, n$. We denote the boundary of $Z_i(r_i)$ by β_i. Each neighborhood $Z_i(r_i)$ admits polar coordinates, and we have

$$\text{Area } Z_i(r_i) = 2\varphi_i(\cosh r_i - 1).$$

For $i = 1, \ldots, n$, we also know that

$$\text{Area } Z_i(r_i) < \text{Area } S.$$

This implies

$$r_i < \operatorname{arccosh}\left(1 + \frac{\text{Area } S}{2\varphi_i}\right).$$

Now

$$\ell(\beta_i) = 2\varphi_i \sinh r_i,$$

so

$$\begin{aligned}
\ell(\beta_i) &< 2\varphi_i \sinh\left(\operatorname{arccosh}\left(1 + \frac{\text{Area } S}{2\varphi_i}\right)\right) \\
&= 2\varphi_i \sqrt{\left(1 + \frac{\text{Area } S}{2\varphi_i}\right)^2 - 1} \\
&< \text{Area } S + 2\varphi_i \\
&< 2\pi(2g - 2 + n).
\end{aligned}$$

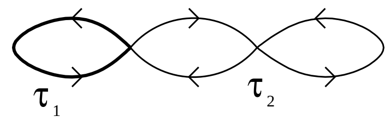

Figure 3.7: β_j as composition of τ_1 and τ_2

Thus the length of the boundary of the neighborhood $Z_i(r_i)$ about p_i, $i = 1, \ldots, n$, is bounded above by $2\pi(2g - 2 + n)$.

Starting from $r_i = 0$, we let r_i grow continuously (for all i simultaneously) until one of the following cases occurs:

1. β_j ceases to be simple for some $j \in \{1, \ldots, n\}$;

2. β_j meets β_k for some $j \neq k$.

Once one of these cases occurs, we fix all the r_i. We will consider each case in turn.

Case 1. We view β_j as the composition of two curves τ_1 and τ_2, which both have initial and final points at the self-intersection of β_j. If β_j has multiple self-intersections, then we choose one and let τ_1 and τ_2 be as in Figure 3.7.

We allow τ_1 and τ_2 to slide in their free homotopy classes on $S \setminus \Sigma$ until they reach their minima. We have

$$\ell(\mathcal{G}(\tau_1)) < \ell(\tau_1) < \ell(\beta_j) < 2\pi(2g - 2 + n),$$

and similarly for $\ell(\mathcal{G}(\tau_2))$. Note that it is possible that one of $\mathcal{G}(\tau_1)$ or $\mathcal{G}(\tau_2)$ is a cone point (but not both, due to the restrictions on the signature of S), in which case the nontrivial geodesic bounds a joker's hat. We cut S open along those $\mathcal{G}(\tau_i)$, $i = 1, 2$, which are not cone points, and remove any resulting V-pieces or joker's hats. Let S^1

be the (possibly empty) remaining connected component. Then

$$\ell(\partial S^1) < 4\pi(2g - 2 + n).$$

Case 2. Suppose that two boundary curves β_j and β_k meet, and that both β_j and β_k are simple (if more than two simple boundary curves meet, choose two). Consider the curve τ obtained by first traversing β_j and then traversing β_k, where the initial and final points of both of these curves are at their intersection point. Note that τ is homotopic to a simple closed curve and is not homotopic to a cone point, as S has signature $(g, n) \neq (0, 3)$. By Proposition 2.6.3 (i), τ is homotopic to a unique simple closed geodesic. We have

$$\ell(\mathcal{G}(\tau)) < \ell(\tau) = \ell(\beta_j) + \ell(\beta_k) < 4\pi(2g - 2 + n).$$

Cutting S open along this geodesic yields at least one joker's hat. Let S^1 denote the (cone-)surface obtained by cutting S open in this way and removing any joker's hats that result. Then

$$\ell(\partial S^1) < 4\pi(2g - 2 + n).$$

We now restart the process; that is, we send out collars $Z_i(r_i)$ from the remaining cone points p_i on S^1 and let r_i grow continuously from $r_i = 0$ until one of the following situations occurs:

1. β_j ceases to be simple for some $j \in \{1, \ldots, n\}$;

2. β_j meets β_k for some $j \neq k$;

3. β_j meets a boundary geodesic γ_i on S^1.

Once one of these cases occurs, we fix all the r_i. Cases 1 and 2 are as above, and if we let S^2 be the (cone-)surface which results from cutting S^1 open along the new geodesics we find and removing all Y-pieces, V-pieces and joker's hats, then

$$\ell(\partial S^2) < 8\pi(2g - 2 + n).$$

Case 3. Consider the curve τ obtained by first traversing β_j and then traversing γ_i, where the initial and final points of both of these curves are at their intersection point. Note that τ is homotopic to a simple curve and is not homotopic to a cone point; if it were, then p_j would live on a joker's hat that would have been removed at the previous step. Thus, τ is homotopic to a unique simple closed geodesic. We have

$$\ell(\mathcal{G}(\tau)) < \ell(\tau) = \ell(\beta_j) + \ell(\gamma_i) < 2\pi(2g - 2 + n) + 4\pi(2g - 2 + n) = 6\pi(2g - 2 + n).$$

Cutting S^1 open along $\mathcal{G}(\tau)$ yields at least one V-piece. Let S^2 be as defined above; then

$$\ell(\partial S^2) < \ell(\partial S^1) + 2\pi(2g - 2 + n) < 6\pi(2g - 2 + n),$$

as $\ell(\mathcal{G}(\tau)) < \ell(\gamma_i) + 2\pi(2g - 2 + n)$ and $\gamma_i \notin \partial M^2$.

We repeat the above process until all of the cone points on S have been removed on V-pieces or joker's hats. Note that at each step, after cutting our cone-surface open along the geodesics we find and removing any Y-pieces, V-pieces and joker's hats, the length of the boundary of the resulting (cone-)surface increases by at most $4\pi(2g - 2 + n)$. To remove all n cone points requires $m \leq 2n$ geodesics and $\mu \leq m$ steps; thus, we have found $\gamma_1, \ldots, \gamma_m$ such that

$$\ell(\partial S^j) < 4\pi j(2g - 2 + n), \quad j = 1, \ldots, \mu$$

and

$$\ell(\gamma_k) < 4\pi k(2g - 2 + n), \qquad k = 1, \ldots, m.$$

To obtain the remaining geodesics in our decomposition, we proceed by induction. That is, we find a suitable simple closed geodesic in the interior of S^μ which is not homotopic to a boundary component, cut S^μ open along this geodesic, remove any Y-pieces, and let $S^{\mu+1}$ be the resulting surface. To find such a geodesic, we create tubular neighborhoods around all boundary geodesics and let the widths grow until a critical case occurs; the area arguments are analogous to those in the induction for compact Riemann surfaces of genus $g \geq 2$ (see [6]). $\square$

Remark 3.3.2. *The proof gives an explicit bound for the length of each partitioning geodesic:*

$$\ell(\gamma_k) < 4\pi k(2g - 2 + n),$$

where γ_k is the kth geodesic in a partition of S. For $L_{g,n}$ we have thus proved the following bound:

$$L_{g,n} < 4\pi(3g - 3 + n)(2g - 2 + n).$$

As another consequence of the proof of Theorem 3.3.1, we have the following statement for admissible cone-surfaces with boundary.

Corollary 3.3.3. *Let S be a compact admissible cone-surface of signature (g, m, n). Then there exists a partition $\mathcal{P} = \{\gamma_1, \ldots, \gamma_{3g-3+m+n}\}$ of S such that*

$$\ell(\gamma_k) < 4\pi k(2g - 2 + n), \quad k = 1, \ldots, 3g - 3 + m + n.$$

Remark 3.3.4. *Buser's proof (see [6]) of Bers' theorem for a compact Riemann surface M finds the initial geodesic(s) in a partition of M in one of two ways. They* *are the geodesics of length* $\leq 2\text{arcsinh}1$*, if any such geodesics exist; these geodesics are known to be pairwise disjoint by the collar theorem for Riemann surfaces. If there are no such geodesics on* M*, then the initial geodesic is one whose length is bounded by an area argument involving the injectivity radius of the surface. The presence of cone points prohibits the use of a similar initial argument for cone-surfaces, as the injectivity radius on a cone-surface tends to zero as we approach any cone point.*

Chapter 4

Parametrizing Admissible Cone-Surfaces

A compact Riemann surface of genus $g \geq 2$ is parametrized by its underlying conformal structure, which can be described using a set of $6g-6$ parameters. As observed in [10, p.85], "Two dimensional hyperbolic cone-manifolds are parametrized by their underlying conformal structures (including position of singular points) and cone angles." Such parametrizations are useful in the study of Teichmüller space (e.g. [6]). We give an explicit description of the necessary parameters in the case of admissible cone-surfaces.

4.1 Twist Parameters

Twist parameters appear when we paste two pairs of pants (or a pair of pants to itself) along two boundary geodesics of the same length. Pasting two Y-pieces gives a compact Riemann surface of signature $(0, 4, 0)$, called an *X-piece*. Pasting two V-pieces results in an admissible cone-surface of signature $(0, 2, 2)$, which we call a

W-piece. Pasting two joker's hats results in an admissible cone-surface of signature $(0,0,4)$, called an *H-piece*. Pasting a Y-piece and a V-piece along two boundary geodesics of the same length yields a *U-piece* (signature $(0,3,1)$), pasting a Y-piece and a joker's hat yields a W-piece, and pasting a V-piece and a joker's hat yields a *Z-piece* (signature $(0,1,3)$).

Let B, B' be two pairs of pants with boundary geodesics γ_i, γ_i' parametrized on $\mathbb{S}^1$, where $i=1,2,3$ if B $[B']$ is a Y-piece, $i=1,2$ if B $[B']$ is a V-piece, and $i=1$ if B $[B']$ is a joker's hat. The parametrizations of the boundary are as given in the pasting conditions. Suppose that $\ell(\gamma_1)=\ell(\gamma_1')$, and fix a real number α. We can then paste γ_1 and γ_1' via the identification

$$\gamma_1(t)=\gamma_1'(\alpha-t)=:\gamma^\alpha(t), \qquad t\in\mathbb{S}^1. \tag{4.1}$$

Here α is the *twist parameter*, and the resulting admissible cone-surface is one of those defined above. We denote by X^α the ordered pair $(\alpha, Y+Y' \bmod (4.1))$, and similarly for the remaining pieces; this is the X-piece *marked* with α.

The following proposition in the case of X-pieces is proved in [6].

Proposition 4.1.1. *Every X-piece [W-piece, H-piece, U-piece, Z-piece] can be obtained by the above construction.*

Proof. Let S denote a given X-, W-, H-, U- or Z-piece. By Proposition 2.6.5, there exists at least one homotopically non-trivial simple closed curve ν on S that is not homotopic to a boundary curve. Replacing ν by the simple closed geodesic in its free homotopy class, then cutting along this geodesic, one obtains a decomposition of S into two pairs of pants. $\square$

4.2 Cubic Pseudographs

Cubic graphs form the combinatorial skeletons of compact Riemann surfaces, and are a useful way of encoding and revealing information about these surfaces. The analogous objects for admissible cone-surfaces will be called cubic pseudographs, as they retain the 3-regular quality of cubic graphs but no longer satisfy the definition of a graph. We will give a brief description of cubic pseudographs, followed by an upper bound on the number of pairwise nonisomorphic cubic pseudographs on admissible cone-surfaces of signature $(g, n) \geq (1, 1)$.

A pseudograph G consists of a set of vertices, a set of edges, and a set of half-edges. There will be three types of vertices in our cubic pseudographs: Y-vertices, V-vertices, and J-vertices. A Y-vertex is a vertex with three emanating half-edges, all of which are free to be glued to other half-edges. It is the combinatorial skeleton for a Y-piece. Similarly, a V-vertex is the combinatorial skeleton for a V-piece. A V-vertex has three emanating half-edges, but only two of these are allowed to be glued to other half-edges. The half-edge which cannot get glued is distinguished by a bar at the end in the figures. Only one of the three half-edges emanating from a J-vertex can be glued to another half-edge; a J-vertex is the skeleton for a joker's hat. An edge in a pseudograph G will consist of the union of two half-edges, and will connect either two distinct vertices or a vertex with itself. Note that since all types of vertices have three emanating half-edges, G will be 3-regular. However, the barred half-edges violate the definition of a graph, in which every half-edge is paired with another to form a full edge. We assume that our pseudographs are connected, i.e. given two distinct vertices v_1 and v_ℓ, there is a sequence of vertices $v_1, v_2, \ldots, v_{\ell-1}, v_\ell$ such that each pair of successive vertices v_i, v_{i+1} in the list is joined by an edge. See Figure 4.1 for examples of cubic pseudographs and their associated cone-surfaces.

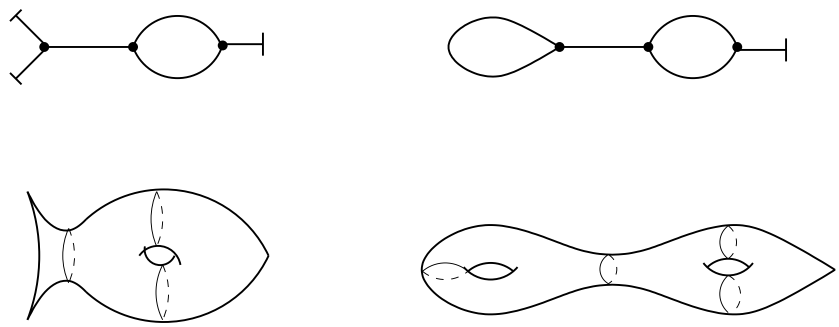

Figure 4.1: Pseudographs and their associated cone-surfaces

Fix a finite cubic pseudograph G. Since G is the combinatorial skeleton for an admissible cone-surface, we know from the collar theorem that G will consist of $2g-2+n$ vertices and will contain $3g-3+n$ edges. These edges correspond to the geodesics needed to decompose an arbitrary admissible cone-surface into pairs of pants.

We denote the vertices of G by $v_1, \ldots, v_{2g-2+n}$; a vertex v_i has three emanating half-edges c_{i1}, c_{i2}, and c_{i3}. If $c_{i\mu}$ and $c_{h\nu}$ are two half-edges which constitute an edge c_k, we write $c_k = (c_{i\mu}, c_{h\nu})$. The half-edges that are not eligible to be glued are labelled e_j, $j = 1, \ldots, n$. So we can fully describe G by the list

$$c_k = (c_{i\mu}, c_{h\nu}),\ e_j,\ k = 1, \ldots, 3g-3+n,\ j = 1, \ldots, n. \tag{4.2}$$

It will sometimes be convenient to view the list (4.2) as the graph. Given a collection $c_{i\mu}, c_{h\nu}, e_j$, we can write a list of ordered triples in which each $c_{i\mu}$, $c_{h\nu}$ and e_j appears exactly once. This list defines a cubic pseudograph; we say that the graph is *admissible* if it is connected.

Definition 4.2.1. *An admissible list as in (4.2) is called a* marked cubic pseudograph.

We say that two marked cubic pseudographs are *isomorphic* if there is a bijection between their vertex sets which preserves adjacency.

The next theorem gives a bound on the number of pairwise nonisomorphic cubic pseudographs of a given size. Note that we restrict to pseudographs which are composed of Y- and V-vertices only. From Theorem 3.1.2, we know that every admissible cone-surface of signature $(g, m, n) > (0, 2, 1)$ can be decomposed into Y- and V-pieces, so that this theorem is of interest. The restriction is for purposes of the proof; we do not know a way to extend the combinatorial arguments to the setting of three types of vertices.

Theorem 4.2.2. *Let $G(g, n)$ denote the number of pairwise nonisomorphic cubic pseudographs with $2g - 2 + n$ Y- and V-vertices, $g > 0, n > 0$, and no J-vertices. Then*

$$G\left(g, n\right) < 1.5 \cdot (2g)^{2g+n-1} \cdot \left(\frac{4g}{e}\right)^{g} \tag{4.3}$$

Proof. We want to construct cubic pseudographs by gluing together a collection of Y- and V-vertices. Since we have at least one cone point on our surface, we can always begin with one V-vertex (which, if the two free half-edges were glued, would be the skeleton of a pointed torus). We can end with a V-vertex or a Y-vertex. We will consider each case in turn.

First assume that we begin with one V-vertex and end with a Y-vertex. We have two free half-edges on the initial piece; attaching a Y-vertex causes the number of free half-edges to increase by 1, while attaching a V-vertex does not change the number of free half-edges. So the number of ways we could have glued our collection of vertices, beginning with a V-vertex and ending with a Y-vertex, to reach this point is at most $2 \cdot 3 \cdots (2g - 1)$, with some $n - 1$ (not necessarily distinct) terms in this product repeated. Now assume that we begin and end with a V-vertex. Then our product looks like $2 \cdot 3 \cdots 2g$, with $n-2$ nonunit exponents. The product

$$2 \cdot 3 \cdots 2g \tag{4.4}$$

with $n-1$ nonunit exponents is at least twice as big as the above products; we take this as our upper bound for the number of distinct possibilities at this point.

The next step is to estimate the sum of the products of $2, 3, \ldots, 2g$ where each product has $2g+n-2$ terms and each number $2, 3, \ldots, 2g$ appears in the product at least once. This number is a function of g and n; denote it by $C(g, n)$. We have that $C(g, n)$ is the coefficient of x^{2g+n-2} in the product

$$(2x+(2x)^2+(2x)^3+\cdots)(3x+(3x)^2+(3x)^3+\cdots)\cdots(2gx+(2gx)^2+(2gx)^3+\cdots), \tag{4.5}$$

since to get a product with $2g+n-2$ terms we take at least one term that contributes a 2, at least one term that contributes a 3, and so on, taking a total of $2g+n-2$ terms. We can rewrite (4.5) as

$$C(g, n) = [x^{2g+n-2}]\frac{2x}{1-2x} \cdot \frac{3x}{1-3x} \cdots \frac{(2g)x}{1-(2g)x}, \tag{4.6}$$

where $[x^{2g+n-2}]$ denotes the coefficient of x^{2g+n-2} in the product. Collecting terms in this equation gives

$$
\begin{aligned}
C(g,n) &= (2g)!\ [x^{2g+n-2}]\frac{x^{2g-1}(1-x)}{(1-x)(1-2x)(1-3x)\cdots(1-(2g)x)} \\
&= (2g)!\ [x^{2g+n-1}]\frac{x^{2g}}{(1-x)(1-2x)(1-3x)\cdots(1-(2g)x)} \\
&\qquad - (2g)!\ [x^{2g+n-1}]\frac{x^{2g}\cdot x}{(1-x)(1-2x)(1-3x)\cdots(1-(2g)x)} \\
&= (2g)!\ (S(2g+n-1,2g) - S(2g+n-2,2g)) \\
&\leq (2g)! S(2g+n-1,2g)
\end{aligned}
$$

where the $S(m,k)$ are the Stirling numbers of the second kind (see [9]). We can estimate the right side by $(2g)! \cdot \frac{(2g)^{2g+n-1}}{(2g)!}$. Hence $C(g,n) \leq (2g)^{2g+n-1}$. At this point, then, there are at most $(2g)^{2g+n-1}$ ways to paste together our set of $2g-2$ Y-vertices and n V-vertices.

Next we need to paste together the remaining free half-edges, of which there are $2g$. Select a free half-edge. There are $(2g-1)$ possible gluing partners for this half-edge. We choose one and glue. There are now $(2g-2)$ free half-edges; we choose one and glue it to one of the remaining $(2g-3)$. Continuing in this manner, we see that there are

$$(2g-1)(2g-3)\cdots 3\cdot 1$$

ways to glue the $2g$ half-edges. Hence

$$G(g,n) \leq (2g)^{2g+n-1} \cdot \frac{(2g-1)!}{2^{g-1}(g-1)!} \tag{4.7}$$

We will now simplify the right-hand side of (4.7). First note that for $g = 1$, the factorial expression is equal to 1. Observe that given any finite number of (indistinguishable) V-vertices and no Y-vertices, there is exactly one way to glue them. So this bound is sharp for $g = 1$. We will henceforth assume that $g > 1$.

In order to simplify the right-hand side of (4.7), we use the version of Stirling's formula which says

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n+\frac{1}{12n}}.$$

This gives

$$(2g)^{2g+n-1} \cdot \frac{(2g-1)!}{2^{g-1}(g-1)!} \leq (2g)^{2g+n-1} \cdot \frac{\sqrt{2\pi(2g-1)}\left(\frac{2g-1}{e}\right)^{2g-1+\frac{1}{12(2g-1)}}}{2^{g-1}\sqrt{2\pi(g-1)}\left(\frac{g-1}{e}\right)^{g-1}} \tag{4.8}$$

For the moment, we will restrict our attention to the last term. We have

$$\begin{aligned}
\frac{\sqrt{2\pi(2g-1)}\left(\frac{2g-1}{e}\right)^{2g-1+\frac{1}{12(2g-1)}}}{2^{g-1}\sqrt{2\pi(g-1)}\left(\frac{g-1}{e}\right)^{g-1}} &= e^{-g} \cdot \frac{(2g-1)^{2g-1/2}}{2^{g-1}(g-1)^{g-1/2}} \cdot \left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}} \\
&= \frac{\sqrt{2}}{e^g} \cdot \frac{(2g-1)^{2g}}{(2g-2)^g} \cdot \sqrt{\frac{2g-2}{2g-1}} \cdot \left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}} \\
&\leq \frac{\sqrt{2}}{e^g} \cdot \frac{(2g)^{2g}}{g^g} \cdot \sqrt{\frac{2g-2}{2g-1}} \cdot \left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}}
\end{aligned}$$

since $2g-1 < 2g$ and $g > 1$ implies $\frac{1}{2g-2} \leq \frac{1}{g}$. Also note that $0 < \sqrt{\frac{2g-2}{2g-1}} < 1$ since $2g-2 < 2g-1$ for all g and $\frac{2g-2}{2g-1} > 0$ for all $g > 1$. We have

$$\begin{aligned}
\frac{\sqrt{2\pi(2g-1)}\left(\frac{2g-1}{e}\right)^{2g-1+\frac{1}{12(2g-1)}}}{2^{g-1}\sqrt{2\pi(g-1)}\left(\frac{g-1}{e}\right)^{g-1}} &\leq \frac{\sqrt{2}}{e^g} \cdot 2^{2g} \cdot \frac{g^{2g}}{g^g} \cdot \left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}} \\
&= g^g \cdot \sqrt{2} \cdot \left(\frac{4}{e}\right)^g \cdot \left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}}
\end{aligned}$$

We now estimate $\left(\frac{2g-1}{e}\right)^{\frac{1}{12(2g-1)}} = \left(\frac{2g-1}{e}\right)^{\frac{1}{24g-12}}$. We seek a constant κ such that

(i) $\kappa \geq \left(\frac{2g-1}{e}\right)^{\frac{1}{24g-12}}$ for $g = 2$

and

(ii) $F(g) = \kappa^{24g-12} - \left(\frac{2g-1}{e}\right)$ is increasing, i.e.

$$\frac{dF}{dg} = \ln\kappa \cdot \kappa^{24g-12} \cdot 24 - \frac{2}{e} \geq 0 \quad \forall g \geq 2$$

or

$$\ln\kappa \cdot \kappa^{24g-12} \geq \frac{1}{12e} \quad \forall g \geq 2$$

We find that $\kappa \approx 1.00274$ satisfies (i). Since $\kappa > 1$, we know that κ^{24g-12} increases with g; condition (ii) reduces to

(ii′) $\ln\kappa \cdot \kappa^{36} \geq \frac{1}{12e}$.

Note that the above value of κ does not satisfy (ii′). By inspection, we find that $\kappa = 1.02$ satisfies both conditions. Thus

$$\frac{\sqrt{2\pi(2g-1)}\left(\frac{2g-1}{e}\right)^{2g-1+\frac{1}{12(2g-1)}}}{2^{g-1}\sqrt{2\pi(g-1)}\left(\frac{g-1}{e}\right)^{g-1}} \leq g^g \cdot 1.02 \cdot \sqrt{2} \cdot \left(\frac{4}{e}\right)^g. \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$(2g)^{2g+n-1} \cdot \frac{(2g-1)!}{2^{g-1}(g-1)!} \leq (2g)^{2g+n-1} \cdot g^g \cdot 1.02 \cdot \sqrt{2} \cdot \left(\frac{4}{e}\right)^g$$

Note that $1.02 \cdot \sqrt{2} \approx 1.4425$, so that

$$(2g)^{2g+n-1} \cdot \frac{(2g-1)!}{2^{g-1}(g-1)!} < 1.5 \cdot (2g)^{2g+n-1} \cdot \left(\frac{4g}{e}\right)^g \tag{4.10}$$

as desired. □

4.3 The Admissible Cone-Surfaces

We now want to construct admissible cone-surfaces of signature (g, n) using marked cubic pseudographs as our underlying combinatorial skeletons. See [6, §3.6] for the construction in the case of compact Riemann surfaces.

Fix a marked cubic pseudograph G with vertices $v_1, \ldots, v_{2g-2+n}$, edges $c_1, \ldots, c_{3g-3+n}$, and half-edges $e_1, \ldots, e_n$, where G is described by the list

$$c_k = (c_{i\mu}, c_{h\nu}),\ e_j,\ k = 1, \ldots, 3g-3+n,\ j = 1, \ldots, n$$

as in (4.2). Then choose

$$\begin{aligned} L &= (\ell_1, \ldots, \ell_{3g-3+n}) \in \mathbb{R}_+^{3g-3+n}, \\ A &= (\alpha_1, \ldots, \alpha_{3g-3+n}) \in \mathbb{R}^{3g-3+n}, \\ N &= (2\varphi_1, \ldots, 2\varphi_n) \in I^n, \end{aligned}$$

where $I = (0, \pi)$, and define an admissible cone-surface $F(G, L, A, N)$ as follows. To each vertex v_i with half-edges c_{i1}, c_{i2}, c_{i3}, we associate a pair of pants with the appropriate number of boundary geodesic(s) $\gamma_{i\mu}$ and of cone angle(s) $2\varphi_j$. All boundary geodesics are parametrized on $\mathbb{S}^1 = \mathbb{R}/[t \mapsto t+1]$, where parametrizations of the boundary are as given in the pasting conditions. We have

$$\ell_k = \ell(\gamma_k) = \ell(\gamma_{i\mu}) = \ell(\gamma_{h\nu}),\ k = 1, \ldots, 3g-3+n.$$

This is possible by Theorems 2.5.3 and 2.5.6, and by the comments following the construction of a joker's hat.

Let P represent a Y-piece, V-piece, or joker's hat. Then we can paste P_i and P_h

together along $\gamma_{i\mu}$ and $\gamma_{h\nu}$ via the identification

$$\gamma_{i\mu}(t) = \gamma_{h\nu}(\alpha_k - t) := \gamma_k(t), \ t \in \mathbb{S}^1. \tag{4.11}$$

After pasting all pairs of pants associated to our fixed marked cubic pseudograph G along the appropriate boundary geodesics, we have

$$F = F(G, L, A, N) = P_1 + \cdots + P_{2g-2+n} \ \mathrm{mod}(4.11).$$

Note that G is a connected graph, so F is connected. Also, the boundary geodesics of our pairs of pants were oriented such that F is orientable. Thus F is an admissible cone-surface of signature (g, n) with cone angles $2\varphi_1, \ldots, 2\varphi_n$.

Definition 4.3.1. *The parameters* (L, A) *are the* Fenchel-Nielsen coordinates *of the admissible cone-surface* $F(G, L, A, N)$.

Note that there is a length parameter and a twist parameter associated to each boundary geodesic $\gamma_1, \ldots, \gamma_{3g-3+n}$; these geodesics will be called the *coordinate geodesics* of F.

Maskit [22] has studied Fenchel-Nielsen coordinates on hyperbolic orbifolds, in the context of connecting Fenchel-Nielsen coordinates on an arbitrary hyperbolic orbifold to matrix generators for the corresponding Fuchsian group. He defines a Fenchel-Nielsen system as the signature (including the orders of the cone points) of the orbifold O, together with a table which lists, for each pair of pants in O, the coordinate geodesics or cone points to which the boundary elements of the pair of pants correspond. The Fenchel-Nielsen coordinates are defined as the lengths and twists about the coordinate geodesics. For our purposes, a Fenchel-Nielsen system on an admissible cone-surface $F(G, L, A, N)$ will consist of the underlying cubic pseudograph G, the cone angles as given by N, and the Fenchel-Nielsen coordinates (L, A). Using such a system, we can prove the following theorem:

Theorem 4.3.2. *Let G be a fixed marked cubic pseudograph with $2g-2+n$ vertices, and let N be a fixed set of n cone angles. Then $F(G, L, A, N)$ runs through all admissible cone-surfaces of genus g with n cone points having cone angles in N.*

The proof of this theorem is a straightforward extension of the proof of the analogous result for compact Riemann surfaces. In order to sketch the proof, we must give the following background (see [6, p. 83]).

Definition 4.3.3. *Let A and B be two topological spaces, and let $\phi_0, \phi_1 : A \to B$ be homeomorphisms. We say that ϕ_0 and ϕ_1 are* isotopic *if there exists a continuous map $\iota : [0,1] \times A \to B$ such that*

1. *$\iota(0, \ ) = \phi_0$;*
2. *$\iota(1, \ ) = \phi_1$; and*
3. *$\iota(s, \ ) : A \to B$ is a homeomorphism for each $s \in [0,1]$.*

In the context of admissible cone-surfaces A and B with a map $\phi : A \to B$, we say that ϕ is a homeomorphism if it is a homeomorphism of the underlying topological spaces of A and B, and sends cone points in A to cone points in B. With these definitions in hand, we can now state

Theorem 4.3.4. *Let $\phi : S \to R$ be a homeomorphism of admissible cone-surfaces, and let $\gamma_1, \ldots, \gamma_N$ be pairwise disjoint, simple closed geodesics on S. Then there exists a homeomorphism ϕ' isotopic to ϕ such that the curves $\phi'(\gamma_1), \ldots, \phi'(\gamma_N)$ are closed geodesics on R.*

The proof of this theorem in the case of Riemann surfaces relies on a theorem of Baer-Zieschang (see [6, p. 411]) and the fact that non-trivial simple closed curves are homotopic to simple closed geodesics. The theorem of Baer-Zieschang holds for surfaces with boundary; one can remove a small neighborhood of each cone point in the cone-surface and perform the isotopies on the resulting surface with boundary. By Proposition 2.6.3, the simple clsoed curves involved are again homotopic to simple closed geodesics. We now sketch the proof of Theorem 4.3.2.

Proof. (Sketch) Fix a base surface $F_0 = F(G, L_0, A_0, N)$. Let S be any admissible cone-surface of genus g with n cone points having cone angles in N. Then there exists a homeomorphism $\phi : F_0 \to S$. By Theorem 4.3.4, we can choose ϕ such that the images $\phi(\gamma_1), \ldots, \phi(\gamma_{3g-3+n})$ of the coordinate geodesics $\gamma_1, \ldots, \gamma_{3g-3+n}$ of F_0 form a system of pairwise disjoint simple closed geodesics on S. Thus ϕ must map pairs of pants on F_0 onto pairs of pants on S. These define S as a surface $S = F(G, L, A, N)$ for appropriate L and A. Note that ϕ is a homeomorphism, so G is the underlying graph of S. $\square$

We conclude our discussion of Fenchel-Nielsen parameters by giving a large family of pairwise non-isometric examples of admissible cone-surfaces. We follow Buser [6, p.84]. Let $\mathfrak{G}$ be the set of all pairwise non-isomorphic marked cubic pseudographs with $2g-2+n$ vertices (where the restrictions on g and n are as for admissible cone-surfaces). We can view each $G \in \mathfrak{G}$ as a list as in (4.2). Fix a set N of n admissible cone-angles. Then, for each fixed $G \in \mathfrak{G}$, we define

$$\begin{aligned}\mathcal{F}(G) \quad = \quad & \{F(G, L, A, N) \mid 0 < \ell_1 < \cdots < \ell_{3g-3+n} < 2\mathrm{arcsinh}(\cos\varphi), \\ & 0 < \alpha_1, \ldots, \alpha_{3g-3+n} < \frac{1}{4}\},\end{aligned}$$

where $L = (\ell_1, \ldots, \ell_{3g-3+n})$, 2φ is the largest cone angle in the set N, and $A = (\alpha_1, \ldots, \alpha_{3g-3+n})$. We define

$$\mathcal{F}_{g,n} = \cup_{G \in \mathfrak{G}} \mathcal{F}(G).$$

The surfaces in $\mathcal{F}_{g,n}$ are pairwise non-isometric.

Proof. Suppose that $F = F(G, L, A, N) \in \mathcal{F}_{g,n}$. We want to recover G, L and A from the intrinsic geometry of F. We begin with the lengths of the coordinate geodesics. By Corollary 3.2.2, the coordinate geodesics $\gamma_1, \ldots, \gamma_{3g-3+n}$ are the only geodesics on F of length less than $2\text{arcsinh}(\cos\varphi)$. Thus, by finding all geodesics on F of length less than $2\text{arcsinh}(\cos\varphi)$ and arranging the lengths of these geodesics in increasing order, we find L. Note that this process also recovers the graph G, as we now know the location and types of the various pairs of pants.

It remains to determine A. Consider a fixed γ_k on F. It is the boundary of one or more pairs of pants of F, and by Propositions 2.5.2, 2.5.5 and 2.6.3, there exist unique simple perpendiculars between γ_k and the other boundary geodesics or cone points of these building blocks. Each pair of these perpendiculars on a given building block decomposes γ_k into two arcs of equal length. By hypothesis, $0 < \alpha_k < \frac{1}{4}$. Thus the minimum distance among the intersection points of the four perpendiculars with γ_k determines α_k. Hence we know A. $\square$

Chapter 5

Orbifolds

We now specialize to the study of cone-manifolds which are orbifolds, that is, whose cone half-angles are of the form $\frac{\pi}{k}$, for k an integer greater than 1.

5.1 Coordinate Charts

In Chapter 2, we studied usual charts and conical charts on cone-surfaces. On orbisurfaces, we can define a variation of these charts which carries more information; namely, it tells us about the local group action at the cone point. In the following definitions and exposition, we consider orbifolds of arbitrary dimension. Good references for this material include Chapter 2 of [10], [32] and [28].

Definition 5.1.1. *Let X be a Hausdorff space, and let U be an open set in X. An* orbifold coordinate chart *over U is a triple $(U, \Gamma\backslash\tilde{U}, \pi)$ such that:*

1. *$\tilde{U}$ is a connected open subset of $\mathbb{R}^n$,*

2. *Γ is a finite group of diffeomorphisms acting on $\tilde{U}$ with fixed point set of codimension at least two, and*

3. *$\pi : \tilde{U} \to U$ is a continuous map which induces a homeomorphism between $\Gamma\backslash\tilde{U}$ and U. We require $\pi \circ \gamma = \pi$ for all $\gamma \in \Gamma$.*

Now suppose that U and U' are two open sets in a Hausdorff space X with $U \subset U'$. Let $(U, \Gamma\backslash\tilde{U}, \pi)$ and $(U', \Gamma'\backslash\tilde{U}', \pi')$ be charts over U and U', respectively.

Definition 5.1.2. *An* injection $\lambda : (U, \Gamma\backslash\tilde{U}, \pi) \hookrightarrow (U', \Gamma'\backslash\tilde{U}', \pi')$ *consists of an open embedding $\lambda : \tilde{U} \hookrightarrow \tilde{U}'$ such that $\pi = \pi' \circ \lambda$, and for any $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ for which $\lambda \circ \gamma = \gamma' \circ \lambda$.*

Note that the correspondence $\gamma \mapsto \gamma'$ defines an injective homomorphism of groups from Γ into Γ'.

Definition 5.1.3. *A* smooth orbifold *$(X, \mathcal{A})$ consists of a Hausdorff space X together with an atlas of charts $\mathcal{A}$ satisfying the following conditions:*

1. *For any pair of charts $(U, \Gamma\backslash\tilde{U}, \pi)$ and $(U', \Gamma'\backslash\tilde{U}', \pi')$ in $\mathcal{A}$ with $U \subset U'$ there exists an injection $\lambda : (U, \Gamma\backslash\tilde{U}, \pi) \hookrightarrow (U', \Gamma'\backslash\tilde{U}', \pi')$.*

2. *The open sets $U \subset X$ for which there exists a chart $(U, \Gamma\backslash\tilde{U}, \pi)$ in $\mathcal{A}$ form a basis of open sets in X.*

Given an orbifold $(X, \mathcal{A})$, we call the topological space X the *underlying space* of the orbifold. Henceforth orbifolds $(X, \mathcal{A})$ will be denoted simply by O. We now give some examples of orbifolds.

1. Let Γ be a group acting properly discontinuously on a manifold M with fixed point set of codimension at least two. Then the quotient space $O = \Gamma\backslash M$ is an orbifold. Since O can be expressed as a global quotient (that is, as a subset of $\mathbb{R}^n$ modulo the action of a discrete group), it is called a *good* or *global* orbifold. If M is a surface, then O is an orbisurface.

2. Consider the orbisurface whose underlying space is the sphere S^2, and which has one cone point. A neighborhood of this cone point is modelled on $\mathbb{Z}_n\backslash\mathbb{R}^2$, where $\mathbb{Z}_n$ is the group of rotations of order n. We call this orbisurface the $\mathbb{Z}_n$-teardrop. Unlike the previous example, the $\mathbb{Z}_n$-teardrop cannot be covered by a manifold and thus is an example of a *bad* orbifold (see [29]).

3. Consider the orbisurface with underlying space S^2 and two cone points, called a football. If both cone points have the same order (that is, they have the same isotropy groups), then this orbisurface can be realized as a global quotient. Otherwise, it is a bad orbisurface.

4. Let Δ be an equilateral triangle in the Euclidean plane. Consider the group of isometries generated by reflections in the lines which contain the three sides of Δ, and let Δ^* be the orientation-preserving subgroup of this group. Then the quotient $\Delta^*\backslash\mathbb{R}^2$ is an orbisurface (often called a turnover) with three cone points. The underlying space of this orbisurface is the sphere S^2, and each of the cone points has isotropy group $\mathbb{Z}_3$.

We will be interested in orbisurfaces which have a hyperbolic structure. The construction of a Riemannian metric on an orbifold is as in the manifold case, with the metric being defined locally via coordinate charts and patched together using a partition of unity. In addition, the metric must be invariant under the local group actions. A smooth orbifold with a Riemannian metric is a *Riemannian orbifold*. Recall that an orbisurface with a hyperbolic metric of constant curvature -1 will be called a *Riemann orbisurface*. Every Riemann orbisurface arises as a global quotient of the hyperbolic plane by a discrete group of isometries (see [29]).

An orbifold O is said to be *locally orientable* if it has an atlas in which every coordinate chart $(U, \Gamma\backslash\tilde{U}, \pi)$ is such that Γ is an orientation-preserving group. If all injections as in Definition 5.1.2 are induced by orientation-preserving maps, then O is *orientable*. If a compact orientable Riemann orbisurface has cone points of order three and higher, it will be called an *admissible Riemann orbisurface*. Cone points of order two are problematic for a variety of reasons, including the possibility that they can be arbitrarily close to each other. Further reasons for excluding them in certain situations will be clear from the relevant proofs in the last chapter.

5.2 Fuchsian Groups

In Beardon [1], a careful study of the geometry of discrete groups is undertaken. In particular, he looks at Fuchsian groups, which may be considered as discrete groups of isometries of the hyperbolic plane. To every Fuchsian group G, we can associate its Dirichlet polygon; a Dirichlet polygon centered at a point $w \in \mathbb{H}$ is the set of all points which are, among all their images under the action of G, closest to w. Beardon proves the following theorem about such a polygon P([1, Thm. 9.3.3]):

Theorem 5.2.1. *The set of side-pairing elements G^* of P generate G.*

The theorem actually holds for an entire class of polygons of which the Dirichlet polygon is a member; we now give the necessary background for the proof. First, note that P is also a *fundamental domain* for G. That is, P is a domain whose boundary has zero area with respect to the hyperbolic metric, and there is some fundamental set F with $P \subset F \subset \tilde{P}$, where $\tilde{P}$ represents the closure of P relative to the hyperbolic plane $\mathbb{H}$. (A fundamental set for G is a subset of the hyperbolic plane that contains exactly one point for every orbit of the group in the plane.) We say that a fundamental domain D is *locally finite* if and only if every compact subset of the hyperbolic plane meets only finitely many images of $\tilde{D}$ under the action of G. The following theorem gives the relationship between a locally finite fundamental domain for G and the generators of G:

Theorem 5.2.2. *Let D be any locally finite fundamental domain for a Fuchsian group G. Then*

$$G_0 = \{g \in G | g(\tilde{D}) \cap \tilde{D} \neq \emptyset\}$$

generates G.

Proof. We follow Beardon [1, Thm. 9.2.7]. Let $\hat{G}$ be the group generated by G_0. Clearly, $\hat{G} \subset G$. It suffices to show that $G \subset \hat{G}$, as then $\hat{G} = G$ and G_0 generates G.

We work in the unit disc Δ. Since D is a fundamental domain for G, for every $z \in \Delta$ there is an element $g \in G$ with $g(z) \in \tilde{D}$. Suppose that $h(z)$ also lies in $\tilde{D}$. Then $h(z) \in \tilde{D}$ and $h(z) \in hg^{-1}(\tilde{D})$, so $hg^{-1} \in G_0$. This gives equality of cosets, i.e. $\hat{G}h = \hat{G}g$. Thus there exists a well-defined map $\phi : \Delta \to G/\hat{G}$ given by

$$\phi(z) = \hat{G}g,$$

where $g(z) \in \tilde{D}$. We now use this map to show that $G \subset \hat{G}$.

Fix $z \in \Delta$. The local finiteness of D implies that there are finitely many images $g_1(\tilde{D}), g_2(\tilde{D}), \ldots, g_m(\tilde{D})$ which contain z, and their union covers an open neighborhood N of z. If $w \in N$, then $w \in g_j(\tilde{D})$ for some j and

$$\phi(w) = \hat{G}(g_j)^{-1} = \phi(z).$$

Thus every $z \in \Delta$ has an open neighborhood N on which ϕ is constant.

In fact, we can easily see that any function ϕ with this property is constant on all of Δ. Endow $\phi(\Delta)$ with the discrete topology. Since ϕ is a continuous function, $\phi(\Delta)$

is connected. Hence $\phi(\Delta)$ contains exactly one point. Thus ϕ is constant on Δ, and $\phi(z) = \phi(w)$ for all $z, w \in \Delta$.

Fix $g \in G$. Let $z \in D$ and $w \in g^{-1}(D)$. Then ϕ being constant on Δ implies

$$\hat{G} = \phi(z) = \phi(w) = \hat{G}g,$$

so that $g \in \hat{G}$. Thus $G \subset \hat{G}$. $\square$

If, in addition to being locally finite, a fundamental domain for a group G is convex, we call it a *convex fundamental polygon* for G. Theorem 5.2.1 holds for all convex fundamental polygons. The proof is as follows:

Proof. We follow Beardon [1]. By Theorem 5.2.2, it suffices to show that if $h(\tilde{P}) \cap \tilde{P} \neq \emptyset$, then $h \in G^*$. Let $w \in \tilde{P} \cap h(\tilde{P})$. Then there exists an open neighborhood N centered at w and elements $h_0(= \mathrm{id}), h_1, h_2, \ldots, h_t$ in G such that $h = h_j$ for some $j \neq 0$, and

$$\begin{aligned} w &\in h_0(\tilde{P}) \cap \cdots \cap h_t(\tilde{P}); \\ N &\subset h_0(\tilde{P}) \cup \cdots \cup h_t(\tilde{P}). \end{aligned}$$

Decreasing the radius of N if necessary, we can assume that N contains no vertices of any $h_j(\tilde{P})$ except possibly w and no sides of any $h_j(\tilde{P})$ other than those that contain w. Since the boundary of P is the union of the sides of P, ∂P in N consists of exactly one side or two distinct sides with common endpoint w. Similarly, the boundary of $h_j(\tilde{P})$ in N, $j = 1, \ldots, t$, consists of exactly one side or two distinct sides with common endpoint w. So we can relabel and require that two consecutive polygons in the list $h_0(P) = P, h_1(P), h_2(P), \ldots, h_t(P), P = h_0(P)$ have a side in common. Thus

$\tilde{P} \cap h_j^{-1} h_{j+1}(\tilde{P})$ is a geodesic segment of positive length, i.e. a side. So $h_{j+1} = h_j g$ for some $g \in G^*$ and h is in the group generated by the side-pairing elements of P. $\square$

Beardon shows ([1, Thm. 9.4.2]) that a Dirichlet polygon P is in fact a convex fundamental polygon for G. Thus every Fuchsian group G is generated by the side-pairing elements of a Dirichlet polygon associated to it.

Note that since an admissible Riemann orbisurface $O = G\backslash\mathbb{H}$ does not contain any cone points of order 2, we do not have to worry about the case that $s = s'$, or equivalently that a vertex is an interior point of a side. Thus we regard the elliptic fixed points of G as vertices of P. Beardon ([1, p.225]) notes that P contains representatives of all conjugacy classes of elliptic elements in G.

5.3 Finite Covers

A Riemann orbisurface can be thought of as a Riemann surface with cone points; it is natural to ask whether there is a nice relationship between Riemann surfaces and Riemann orbisurfaces. We have the following result:

Proposition 5.3.1. *Every compact Riemann orbisurface is finitely covered by a compact Riemann surface.*

Our proof will rely on the following lemma of Selberg [30]:

Lemma 5.3.2. *Let H be a finitely generated group of n by n matrices (they need not be real, nor must H be discrete). Then H has a normal subgroup of finite index which contains no element of finite order other than the identity.*

Using this lemma, Proposition 5.3.1 is stated as a corollary to a version of Poincaré's polyhedron theorem in [10]; Jim Davis, Chris Judge and Kevin Pilgrim

[20] used a similar method, but without invoking Selberg's lemma, to prove the result independently. Our proof of Proposition 5.3.1 differs from both of the above approaches.

Proof. Let O be a compact Riemann orbisurface. Then we can write $O = \Gamma\backslash\mathbb{H}$, where Γ is a subgroup of $SL(2,\mathbb{R})$. The compactness of O implies that Γ is finitely generated (see [2]). For there exists a compact subset $K \subset \mathbb{H}$ such that $\Gamma.K = \mathbb{H}$. Since the action of Γ on $\mathbb{H}$ is properly discontinuous, the set

$$E = \{\gamma \in \Gamma |\ \gamma.K \cap K \neq \emptyset\}$$

is finite. By Lemma 6.6 of [3], E is a generating set for Γ.

Now Lemma 5.3.2 implies that Γ has a normal subgroup Γ_0 of finite index which contains no elliptic elements. Thus $\Gamma_0\backslash\mathbb{H}$ is a compact Riemann surface, and $\Gamma_0\backslash\mathbb{H}$ finitely covers $\Gamma\backslash\mathbb{H}$. $\square$

It is natural to ask whether there is a unique "smallest" cover of a compact Riemann orbisurface by a compact Riemann surface. This is an open question.

Chapter 6

Spectral Geometry of Orbifolds

6.1 The Laplace Spectrum

Our goal is to study the spectrum of the Laplace operator as it acts on smooth functions on an orbifold. First, we must define what is meant by a smooth function.

Definition 6.1.1. *Let O be a compact Riemannian orbifold. A map $f : O \to \mathbb{R}$ is a* smooth function *on O if for every coordinate chart $(U, \Gamma\backslash\tilde{U}, \pi)$ on O, the lifted function $\tilde{f} = f \circ \pi$ is a smooth function on $\tilde{U}$.*

If O is a compact Riemannian orbifold and f is a smooth function on O, then we define the Laplacian Δf of f by lifting f to local covers. That is, we lift f to $\tilde{f} = f \circ \pi$ via a coordinate chart $(U, \Gamma\backslash\tilde{U}, \pi)$. We denote the Γ-invariant metric on $\tilde{U}$ by g_{ij} and set $\rho = \sqrt{det(g_{ij})}$. Then we can define

$$\Delta\tilde{f} = \frac{1}{\rho}\sum_{i,j=1}^{n}\frac{\partial}{\partial\tilde{x^i}}(g^{ij}\frac{\partial f}{\partial\tilde{x^j}}\rho).$$

We are really interested in the eigenvalues of the Laplace operator as it acts on smooth functions. In analogy with the manifold case, Chiang ([8]) proved the following theorem:

Theorem 6.1.2. *Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*

2. *Each eigenvalue λ_i has finite multiplicity.*

3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta\phi_i = \lambda_i \phi_i$.*

The multiplicity of the ith eigenvalue λ_i is the dimension of the space of eigenfunctions with eigenvalue λ_i.

6.2 The Heat Equation

Much information about the relationship between the Laplace spectrum of an orbifold O and the geometric properties of O can be gleaned by studying the heat equation:

$$\Delta F = -\frac{\partial F}{\partial t},$$

where $F(x,t)$ is the heat at a point $x \in O$ at time t.

With initial data $f : O \to \mathbb{R}$, $F(x,0) = f(x)$, a solution of the heat equation is given by

$$F(x) = \int_O K(x,y,t) f(y) dy.$$

Here $K : O \times O \times \mathbb{R}_+^* \to \mathbb{R}$ is a C^∞ function given by the convergent series

$$K(x, y, t) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y). \tag{6.1}$$

The eigenfunctions ϕ_i of Δ are chosen such that they form an orthonormal basis of $L^2(O)$, the square-integrable functions on O. We say that K is the *fundamental solution of the heat equation* on O, or the *heat kernel* on O. The appropriate physical interpretation is that $K(x, y, t)$ is the temperature at time t at the point y when a unit of heat (a Dirac delta-function) is placed at the point x at time $t = 0$.

By considering the asymptotic behavior of K as $t \to 0$, we can recover information about the geometry of O. In this direction, Farsi showed (see [14]) that Weyl's asymptotic formula can be extended to orbifolds. In particular, she proved

Theorem 6.2.1. *Let O be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ we have*

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O) \frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$. Here $B_0^n(1)$ denotes the n-dimensional unit ball in Euclidean space.

This theorem implies that, in analogy with the manifold case, the Laplace spectrum determines an orbifold's dimension and volume.

By looking at the terms of the asymptotic expansion of the trace of the heat kernel, Gordon, Greenwald, Webb and Zhu ([17]) have given the following obstruction to isospectrality:

Theorem 6.2.2. *Let O be a Riemannian orbifold with singularities. If M is a manifold such that O and M have a common Riemannian cover, then M and O cannot be isospectral.*

In particular, this implies that a hyperbolic orbifold with singularities is never isospectral to a hyperbolic manifold.

We saw in Proposition 5.3.1 that every compact Riemann orbisurface is finitely covered by a compact Riemann surface. A natural (and open) question is: Do isospectral compact Riemann orbisurfaces have a common finite cover?

6.3 Obstructions to Isospectrality

We want to investigate further obstructions to isospectrality; our focus will be the case of orbisurfaces. In analogy with the surface case, we can define the Euler characteristic and state a Gauss-Bonnet theorem for orbisurfaces (see [33]).

Definition 6.3.1. *Let O be an orbisurface with s cone points of orders $m_1, \ldots, m_s$. Then we define the (orbifold) Euler characteristic of O to be*

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}),$$

where $\chi(X_O)$ is the Euler characteristic of the underlying (topological) space of O.

The Gauss-Bonnet theorem for orbifolds gives the usual relationship between topology and geometry.

Theorem 6.3.2. *(Gauss-Bonnet) Let O be a Riemannian orbisurface. Then*

$$\int_O K dA = 2\pi\chi(O),$$

where K is the curvature and $\chi(O)$ is the orbifold Euler characteristic of O.

Note that we define the curvature of an orbifold O at a point $x \in O$ with coordinate chart $(U, \Gamma\backslash\tilde{U}, \pi)$ to be the curvature at a lift $\tilde{x} \in \tilde{U}$ of x.

Combining the Gauss-Bonnet theorem with Weyl's asymptotic formula, we see that for an orbisurface with given curvature, the spectrum determines the orbifold Euler characteristic. However, since the orbifold Euler characteristic involves both the genus of the underlying surface and the orders of the cone points in the orbisurface, it is not immediately clear that the spectrum determines the genus. This is still an open question.

In the case of orientable orbisurfaces, Gordon, Greenwald, Webb and Zhu ([17]) show that the Euler characteristic can be recovered from the asymptotic expansion of the trace of the heat kernel. Together with some computations for cone points, this allows them to define a spectral invariant which determines whether an orbifold is a football or teardrop and determines the orders of the cone points. In a similar vein, we give the following obstructions to isospectrality.

Proposition 6.3.3. *Fix $g \geq 1$ and $m \geq 2$. Let O be a compact orientable Riemann orbisurface of genus g with exactly one cone point of order m. Let O' be in the class of compact hyperbolic orientable orbifolds of genus g with cone points of order 2 and higher, and suppose that O is isospectral to O'. Then O' must be an orbisurface with exactly one cone point, and its order is also m.*

Proof. Since O is isospectral to O', Theorem 6.2.1 implies that O' is two-dimensional. By Theorem 6.2.2, O' must contain at least one cone point. We have $\chi(X_O) = \chi(X_{O'})$ by hypothesis, and the observation following Theorem 6.3.2 implies that $\chi(O) = \chi(O')$.

Suppose that O' has one cone point of order n_1. It follows that

$$\frac{1}{m} = \frac{1}{n_1},$$

or $m = n_1$. Now suppose that O' has two cone points of orders n_1 and n_2. Then

$$\frac{1}{m} + 1 = \frac{1}{n_1} + \frac{1}{n_2}.$$

But $n_i \geq 2$ for $i = 1, 2$, so $\frac{1}{n_1} + \frac{1}{n_2} \leq 1$. This is a contradiction, hence O and O' are not isospectral. This argument is easily extended to the case when O' is assumed to have more than two cone points. $\square$

We can extend Proposition 6.3.3 to the case of two orbifolds with different underlying spaces.

Proposition 6.3.4. *Let O be a compact orientable Riemann orbisurface of genus $g_0 \geq 0$ with k cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be a compact hyperbolic orientable orbifold of genus $g_1 \geq g_0$ with l cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k + h)$, then O is not isospectral to O'.*

Proof. Suppose O is isospectral to O'. As in the preceding proof, we have that O' must be two-dimensional and contain at least one cone point. Also, $\chi(O) = \chi(O')$, i.e.

$$2 - 2g_0 - k + \frac{1}{m_1} + \cdots + \frac{1}{m_k} = 2 - 2g_1 - l + \frac{1}{n_1} + \cdots + \frac{1}{n_l}.$$

So

$$\frac{1}{m_1} + \cdots + \frac{1}{m_k} = 2(g_0 - g_1) + k - l + \frac{1}{n_1} + \cdots + \frac{1}{n_l}$$

or equivalently

$$\frac{1}{m_1} + \cdots + \frac{1}{m_k} \leq h + k - \frac{l}{2}.$$

But $k + h \leq \frac{l}{2}$ by hypothesis, which implies that the sum of the reciprocals of the orders of the cone points $m_1, \ldots, m_k$ is nonpositive. This is a contradiction. $\square$

If we assume that $h = 0$, i.e. that $g_0 = g_1$, then we have the following special case of Proposition 6.3.4.

Corollary 6.3.5. *Fix $g \geq 0$. Let O be a compact orientable Riemann orbisurface of genus g with k cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be a compact hyperbolic orientable orbifold of genus g with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then O is not isospectral to O'.*

Note that in all of the above results, we have the hypothesis that O' is hyperbolic. In the usual case of surfaces, we know that any surface isospectral to a given one of fixed constant curvature must have the same constant curvature. The proof of this uses the asymptotic expansion of the trace of the heat kernel; in the case of orbisurfaces, this expansion is more complicated and it is no longer clear that fixed constant curvature is a spectral invariant.

6.4 Huber's Theorem

Huber's theorem is a powerful tool in the study of questions of isospectrality of compact Riemann surfaces. It allows us to translate information about eigenvalues into information about the geometry of the surface, and specifically about the lengths of closed geodesics on the surface (see [6]).

Theorem 6.4.1. *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

The length spectrum is the sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order.

The idea of the proof is as follows. First, a fundamental domain argument leads to a length trace formula. The known eigenfunction expansion of the heat kernel as given in (6.1) is then plugged into this length trace formula to obtain the Selberg Trace Formula. The Selberg Trace Formula contains information about the eigenvalues on one side and information about the lengths of closed geodesics and the area of the Riemann surface on the other; it is then a matter of showing that the eigenvalues determine the lengths of the closed geodesics and vice versa. Further background and the case of the Selberg Trace Formula for compact hyperbolic manifolds can be found in Randol (in [7]). Other sources for the Riemann surface case include [6] and [27].

We want to extend Huber's theorem to the class of admissible Riemann orbisurfaces. To begin, we need to exhibit a Selberg Trace Formula for such objects. We initially follow the development of Randol (in [7]) and Pitkin [27], then, following Hejhal [19], comment on the necessary modifications in the case of admissible Riemann orbisurfaces. We begin with some preliminaries.

Definition 6.4.2. *A function $f(z,w) : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be* point-pair invariant *if it only depends on the hyperbolic distance between z and w.*

Let $k(z,w) = k(d(z,w)) \in C_c^\infty(\mathbb{R})$ be an even function which is point-pair invariant. Then we can define an integral operator $\mathcal{K}$ on $C^\infty(\mathbb{H})$ by

$$\mathcal{K}f(z) = \int_{\mathbb{H}} k(z,w) f(w) d\mu(w), \tag{6.2}$$

where $d\mu(w)$ is the usual area element on the upper-half plane with the Poincaré metric. Arguments involving radial eigenfunctions and the radialization of an arbitrary function give the following relationship between $\mathcal{K}$ and Δ:

Proposition 6.4.3. *Let ϕ be any eigenfunction of Δ on $\mathbb{H}$, with $\Delta\phi = \lambda\phi$. Then ϕ is also an eigenfunction of any integral operator $\mathcal{K}$ which is generated by a point-pair* *invariant $k(z,w)$. Furthermore, there is a function h such that*

$$\mathcal{K}\phi = h(\lambda)\phi.$$

The point here is that $h(\lambda)$ has no dependence on ϕ, which means that we can choose any eigenfunction of Δ with eigenvalue λ and use it to compute $h(\lambda)$. Fix $s \in \mathbb{C}$ and consider the function $y^s = e^{s \log y}$. Since the Laplacian in the upper half-plane model for $\mathbb{H}$ is given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

the reader can check that

$$\Delta y^s = s(1-s) y^s;$$

that is, y^s is an eigenfunction of the Laplacian with eigenvalue $s(1-s)$. Note that $s(1-s)$ varies over $\mathbb{C}$ as s does. Setting $s = \frac{1}{2} + ir$, we see that if $\lambda = s(1-s)$ is an eigenvalue, then $\lambda = r^2 + \frac{1}{4}$. Thus we can think of h as a function of r.

Now consider a function $f \in L^2(\Gamma\backslash\mathbb{H})$. We can lift f to an automorphic function on $\mathbb{H}$, meaning that $f(\gamma z) = f(z)$ for $\gamma \in \Gamma$. Note that f is L^2 on compact subsets of $\mathbb{H}$. Conversely, if we have a function on $\mathbb{H}$ which is L^2 on compact sets and automorphic with respect to Γ, then we can view f as an L^2 function on $\Gamma\backslash\mathbb{H}$.

Taking $k \in C_c^\infty(\mathbb{R})$ to be an even point-pair invariant kernel as before, we can define

$$K(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w),$$

which is a finite sum for specified z and w. It can be shown that $K(z,w)$ is a symmetric kernel, and if $\tilde{\mathcal{K}}$ is the integral operator on $L^2(\Gamma\backslash\mathbb{H})$ generated by K, then the eigenfunctions of Δ are also eigenfunctions of $\tilde{\mathcal{K}}$. That is,

Proposition 6.4.4. *Let $\phi_0, \phi_1, \ldots$ be an orthonormal basis of eigenfunctions for Δ, with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Then*

$$\tilde{\mathcal{K}}\phi_n = h(r_n)\phi_n, \quad \forall n = 0, 1, \ldots.$$

Recall that $\lambda_n = \frac{1}{4} + r_n^2$. As a corollary of this proposition, we can find an expression for $K(z,w)$ in terms of h and the eigenfunctions ϕ_n. In fact, we can replace $k(z,w)$ by a more general function $L(z,w) = L(Q(z,w))$, where $L(\rho) \in C_c^\infty([0,\infty))$ and

$$Q(z,w) = \frac{|z-w|^2}{\Im z \Im w}.$$

Corollary 6.4.5. *We have*

$$\sum_{\gamma \in \Gamma} L(z, \gamma w) = \frac{1}{2} \sum_n h(r_n)\phi_n(z)\phi_n(w) \tag{6.3}$$

in the L^2-sense, where all roots r_n are counted in the sum on the right side.

It can further be shown that this expansion is valid pointwise, and that the sum on the right side converges uniformly and absolutely. By setting $w = z$ and integrating over a nice fundamental domain for $\Gamma\backslash\mathbb{H}$, we get the expression

$$\sum_n h(r_n) = 2 \sum_{\gamma \in \Gamma} \int_{\mathcal{F}} L(z, \gamma z) d\mu.$$

Recall that such a nice fundamental domain, with sides identified in pairs and elliptic elements represented by vertices, exists ([1]).

We obtain the Selberg Trace Formula by expanding this integral term-by-term, where the expansion is over the different types of possible conjugacy classes (in our case, trivial, elliptic, and hyperbolic). In this expansion, the Fourier transform of

$h(r)$, given by

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iru} dr,$$

will appear. We can state the Selberg Trace Formula in terms of functions $h(r)$ and $g(u)$ which are not required to have compact support; namely, $h(r)$ satisfies the following (weaker) conditions:

Assumption 6.4.6.

- *$h(r)$ is an analytic function on $|Im(r)| \leq \frac{1}{2} + \delta$;*
- *$h(-r) = h(r)$;*
- *$|h(r)| \leq M[1 + |Re(r)|]^{-2-\delta}$.*

The numbers δ and M are some positive constants.

Hejhal [19] obtains the following version of the Selberg Trace Formula for the case of interest:

Theorem 6.4.7. *Suppose that*

- *$\Gamma \subset PSL(2, \mathbb{R})$ is a Fuchsian group with compact fundamental region;*
- *$h(r)$ satisfies Assumption 6.4.6;*
- *$\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal eigenfunction basis for $L^2(\Gamma \backslash \mathbb{H})$.*

Then

$$\begin{aligned} \sum_{n=0}^{\infty} h(r_n) &= \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr \\ &+ \sum_{\substack{\{R\} \\ elliptic}} \frac{1}{2m(R) \sin\theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R) r}}{1 + e^{-2\pi r}} h(r) dr \\ &+ \sum_{\substack{\{P\} \\ hyperbolic}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)], \end{aligned} \tag{6.4}$$

where all the sums and integrals in sight are absolutely convergent.

The first and last terms on the right side of (6.4) match the terms which appear in the Selberg Trace Formula for compact Riemann surfaces, where $\mu(F)$ is the area of a fundamental region of Γ, $N(P)$ is the norm of a hyperbolic element P and P_c is a primitive hyperbolic element with $P = P_c^k$ for some $k \geq 1$. Every hyperbolic element P in $PSL(2, \mathbb{R})$ is conjugate to a dilation $z \mapsto mz, m > 1$; we call m the *norm* of P and denote it by $N(P)$. A *primitive* hyperbolic element is one which cannot be written as a nontrivial power of another hyperbolic element. Note that ([19, Prop. 2.3])

$$\inf_{z \in \mathbb{H}} d(z, Tz) = \ln N(T),$$

where d is the distance on $\mathbb{H}$. The infimum is realized by all points z which lie on the geodesic in $\mathbb{H}$ which is invariant under the action of T. In the sum over elliptic conjugacy classes, $m(R)$ denotes the order of the centralizer (in Γ) of a representative R and $\theta(R)$ represents the angle of rotation. We have $Tr(R) = 2\cos\theta(R)$, and $0 < \theta < \pi$.

In the proof of our partial extension of Huber's theorem, we will need the following results of Stanhope [32].

Theorem 6.4.8. *Let $\mathcal{S}$ be a collection of isospectral orientable compact Riemannian orbifolds that share a uniform lower bound $\kappa(n-1)$, κ real, on Ricci curvature. Then there are only finitely many possible isotropy types, up to isomorphism, for points in an orbifold in $\mathcal{S}$.*

Theorem 6.4.9. *Let isol$\mathcal{S}$ be a collection of isospectral Riemannian orbifolds with only isolated singularities that share a uniform lower bound $\kappa \in \mathbb{R}$ on sectional curvature. Then there is an upper bound on the number of singular points in any orbifold, O, in isol$\mathcal{S}$ depending only on Spec(O) and κ.*

We are now prepared to state our partial extension of Huber's theorem to the setting of admissible Riemann orbisurfaces.

Theorem 6.4.10. *If two admissible Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Proof. We will consider Theorem 6.4.7 for a specific function $h(r)$. Fix $t > 0$ and let $h(r) = e^{-r^2 t}$. Then $h(r)$ satisfies Assumption 6.4.6, and we have

$$\begin{aligned} g(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)e^{-iru}dr \\ &= \frac{1}{\sqrt{4\pi t}}e^{-u^2/4t} \end{aligned}$$

where the first line is the definition of $g(u)$ as the Fourier transform of $h(r)$ and the second line follows from Fourier analysis using a standard polar coordinates trick.

By Theorem 6.4.7, we have

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-r_n^2 t} &= \frac{\mu(F)}{4\pi}\int_{-\infty}^{\infty} re^{-r^2 t}\tanh(\pi r)dr \\ &+ \sum_{\substack{\{R\} \\ \text{elliptic}}} \frac{1}{2m(R)\sin\theta(R)}\int_{-\infty}^{\infty}\frac{e^{-2\theta(R)r}}{1+e^{-2\pi r}}e^{-r^2 t}dr \\ &+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}}\frac{1}{\sqrt{4\pi t}}e^{-(\ln N(P))^2/4t}. \end{aligned} \tag{6.5}$$

Let O and O' be admissible Riemann orbisurfaces with the same Laplace spectrum. By Theorem 6.2.1, the Laplace spectrum determines an orbifold's volume. So we must have $\text{vol}(O) = \text{vol}(O')$, and thus the first term on the right side of (6.5) must be the same for O and O'.

Note that both O and O' have a metric of constant curvature -1, and hence share a uniform lower bound on their Ricci curvature and on their sectional curvature. By Theorem 6.4.8, we know that there are only finitely many possible isotropy types, up to isomorphism, for points in O $[O']$. By Theorem 6.4.9, there is an upper bound on the number of cone points in O $[O']$. That is, there are only finitely many cone points in O $[O']$. Putting these two facts together, we see that we can determine the sum over the elliptic elements in (6.5) up to finitely many possibilities. That is, up to finitely many possibilities, we know the function

$$f(t) = \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} e^{-(\ln N(P))^2/4t}.$$

Consider the function $f(t)e^{\omega^2/4t}$. Take the limit of this function as $t \downarrow 0$; we see that there is a unique $\omega > 0$ for which this limit is finite and nonzero. Let γ_1 be the shortest primitive closed geodesic in O. Then $\omega = \ell(\gamma_1)$. We remove the contribution of γ_1 and all its powers from $f(t)$, and proceed as above to find the length of the next-shortest primitive closed geodesic. In this way, we can determine the lengths of the hyperbolic elements in O $[O']$, up to finitely many possible lists of lengths.

Now suppose we know the length spectrum and the orders of the cone points for O $[O']$. The argument that we then know the Laplace spectrum of O $[O']$ is exactly as for Riemann surfaces (see [27, p. 45]). We include it for completeness.

First, we multiply both sides of (6.5) by $e^{-t/4}$ and recall that $\lambda_n = -\frac{1}{4} - r_n^2$ to obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} e^{\lambda_n t} &= \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} r e^{-r^2 t} \tanh(\pi r) dr \\
&+ \sum_{\substack{\{R\} \\ \text{elliptic}}} \frac{e^{-t/4}}{2m(R)\sin\theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1+e^{-2\pi r}} e^{-r^2 t} dr \\
&+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} \cdot \frac{e^{-t/4}}{\sqrt{4\pi t}} e^{-(\ln N(P))^2/4t}. \qquad (6.6)
\end{aligned}
$$

Knowledge of the length spectrum and the orders of the cone points in (6.6) translates to knowledge of the function

$$
\begin{aligned}
c(t) &= \sum_{n=0}^{\infty} e^{\lambda_n t} - \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} r e^{-r^2 t} \tanh(\pi r) dr \\
&= \sum_{-\frac{1}{4} \le \lambda_n < 0} e^{\lambda_n t} - \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} r e^{-r^2 t} \tanh(\pi r) dr + \sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t} \\
&= \sum_{-\frac{1}{4} \le \lambda_n < 0} e^{\lambda_n t} - \sigma(t) e^{-t/4} \mu(F) + \sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t}, \qquad (6.7)
\end{aligned}
$$

where

$$
\sigma(t) = \frac{1}{2\pi} \int_0^{\infty} e^{-r^2 t} r \tanh(\pi r) dr.
$$

Note that as $t \to \infty$, $\sigma(t) \to 0$. For

$$
\begin{aligned}
\sigma(t) &= \frac{1}{2\pi}\int_0^\infty e^{-r^2 t} r \tanh(\pi r) dr \\
&= \frac{1}{2\pi}\int_0^\infty e^{-r^2 t} r \frac{e^{2\pi r}-1}{e^{2\pi r}+1} dr \\
&\leq \frac{1}{2\pi}\int_0^\infty e^{-r^2 t} r \left(e^{2\pi r}-1\right) dr \\
&\leq \frac{1}{2\pi}\int_0^\infty e^{-r^2 t} r e^{2\pi r} dr \\
&\leq \frac{1}{2\pi} e^{2\pi}\int_0^1 e^{-r^2 t} r dr + \frac{1}{2\pi}\int_1^\infty e^{-r^2 t} r e^{2\pi r^2} dr \\
&= \frac{1}{4\pi} e^{2\pi}\int_0^1 e^{-ut} du + \frac{1}{4\pi}\int_1^\infty e^{u(2\pi - t)} du
\end{aligned}
$$

where we substitute $u = r^2$. Then

$$
\sigma(t) \leq \frac{1}{4\pi} e^{2\pi}\left(-\frac{e^{-t}}{t} + \frac{1}{t}\right) + \frac{1}{4\pi}\left[\frac{e^{u(2\pi - t)}}{2\pi - t}\right]_1^\infty .
$$

As $t \to \infty$, both of these terms go to zero. Thus, as $t \to \infty$, $\sigma(t) \to 0$.

If $\lambda_1 \geq -\frac{1}{4}$, then $-\lambda_1$ is the unique $\omega > 0$ such that

$$
0 < \lim_{t \to \infty} e^{\omega t} c(t) < \infty
$$

In fact, this limit is the multiplicity m_1 of λ_1. We can therefore subtract $m_1 e^{\lambda_1 t}$ from $c(t)$ and continue in this way to find all the small eigenvalues. Once all the small eigenvalues have been found, the function

$$
\tilde{c}(t) = -\sigma(t) e^{-t/4} \mu(F) + \sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t}
$$

has the property that for $\omega > 0$,

$$
\lim_{t \to \infty} e^{\omega t} \tilde{c}(t)
$$

is 0 or ∞. So we can now multiply $\tilde{c}(t)$ by $-\frac{e^{t/4}}{\sigma(t)}$ and take the limit as $t \to \infty$ to get $\mu(F)$. We then know the function

$$\sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t},$$

and we can determine the remaining eigenvalues in the same way as we found the small eigenvalues. Hence the spectrum of the Laplacian is determined by the length spectrum and the orders of the cone points, and the proof is complete. □

6.5 Finiteness of Isospectral Sets

McKean [24] showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of admissible Riemann orbisurfaces. Specifically, we show

Theorem 6.5.1. *Let O be an admissible Riemann orbisurface of genus $g \geq 1$. In the class of compact orientable hyperbolic orbifolds with cone points of order three and higher, there are only finitely many members which are isospectral to O.*

Remark 6.5.2. *Note that there is no need for a dimension restriction on the orbifolds that can be isospectral to O, by Theorem 6.2.1. In addition, by Theorem 6.2.2, there can be no Riemann surfaces isospectral to O.*

We begin with some preliminaries. Let G be a subgroup of $SL(2, \mathbb{R})$. Then $h \in G$ can be represented by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$. We define the *trace* of h to be

$$tr(h) = a + d,$$

i.e. the sum of the diagonal elements in the matrix that represents h.

McKean [24] states the following proposition, which he attributes to Fricke and Klein [15]. We record the proof here for completeness.

Proposition 6.5.3. *Let $M = G\backslash\mathbb{H}$ be a Riemann surface of genus $g \geq 2$, where $G \leq SL(2, \mathbb{R})$. Let the set $\{h_1, \ldots, h_n\}$, $n \leq 2g$, be a generating set for G. Then the single, double, and triple traces*

$$\begin{aligned} &tr(h_i), \\ &tr(h_i h_j), \quad i < j \\ &tr(h_i h_j h_k), \quad i < j < k \end{aligned}$$

determine G up to a conjugation in $PSL(2, \mathbb{R})$ or a reflection.

Proof. Let G and G' be two subgroups of $SL(2, \mathbb{R})$ with the same single, double, and triple traces of their generators. Fix $h_1 \in G$. Since the single traces of the generators of G and G' are equal, we can pair h_1 with an element in G' that translates the same amount; that is, we can suppose that $h_1 = h_1'$ and that $h_1(z) = m^2 z$ with $m > 1$. Note that any other diagonal element in G fixes the same geodesic in $\mathbb{H}$ as h_1 and is thus a multiple of h_1. So we can assume that h_1 is the only diagonal element in $\{h_1, \ldots, h_n\}$. For $i > 1$, we have

$$tr(h_i) = a_i + d_i = tr(h_i') = a_i' + d_i'. \tag{6.8}$$

Also,

$$tr(h_1h_i) = tr\left(\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right) = tr\left(\begin{pmatrix} ma_i & mb_i \\ m^{-1}c_i & m^{-1}d_i \end{pmatrix}\right) = ma_i + m^{-1}d_i$$

and similarly for $tr(h_1'h_i')$, so that

$$ma_i + m^{-1}d_i = tr(h_1h_i) = tr(h_1'h_i') = ma_i' + m^{-1}d_i', \tag{6.9}$$

or equivalently

$$m(a_i - a_i') = m^{-1}(d_i' - d_i). \tag{6.10}$$

From equation (6.8) we see that $a_i - a_i' = d_i' - d_i$. But we assumed $m > 1$, so we must have

$$a_i = a_i' \text{ and } d_i = d_i'. \tag{6.11}$$

We also know that $\det(h_i) = \det\ (h_i') = 1$ for all i, so

$$b_ic_i = b_i'c_i' \tag{6.12}$$

for all i. Straightforward calculations show that

$$tr(h_ih_j) - tr(h_i'h_j') = b_ic_j - b_i'c_j' + c_ib_j - c_i'b_j' \tag{6.13}$$

and

$$tr(h_1h_ih_j) - tr(h_1'h_i'h_j') = m(b_ic_j - b_i'c_j') + m^{-1}(c_ib_j - c_i'b_j') \tag{6.14}$$

for $1 < i < j$. Combining equations (6.13) and (6.14) as we combined (6.8) and (6.10), we see that $b_ic_j = b_i'c_j'$ for $1 < i < j$. We want to see that none of these numbers are zero. Suppose $c_2 = 0$. Then

$$h_1^{-n} h_2 h_1^n(\sqrt{-1}) = \frac{a_2\sqrt{-1} + m^{-2n} b_2}{d_2},$$

where this is the Möbius action of $SL(2, \mathbb{R})$ on $\mathbb{H}$. So we get infinitely many images of $\sqrt{-1}$ accumulating at $\frac{a_2}{d_2}\sqrt{-1} \in \mathbb{H}$ (unless $b_2 = 0$, which implies that h_2 is diagonal, contradicting our assumption that h_1 is the only diagonal element in $\{h_1, \ldots, h_n\}$). This contradicts the assumption that G acts properly discontinuously on $\mathbb{H}$. A similar argument with $b_2 = 0$ shows that the off-diagonal entries in the matrix representing the element h_2 are nonzero. Our choice of h_2 was arbitrary, thus none of the off-diagonal entries in the matrices representing the elements $h_2, \ldots, h_n$ and $h'_2, \ldots, h'_n$ are zero. We have

$$\frac{c'_j}{c_j} = \frac{b_i}{b'_i} = \frac{c'_i}{c_i},$$

where the second equality is equation (6.12), and this common ratio is independent of $i > 1$. Since the traces do not tell us whether the ratio is positive, we must allow the reflection

$$G \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose that the common ratio is equal to t^2, i.e.

$$\frac{b_i}{b'_i} = t^2 = \frac{c'_i}{c_i}$$

for all $i > 1$. Then $b_i = t^2 b'_i$ and $c_i = t^{-2} c'_i$ for $i > 1$. Thus

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} a_i & t^2 b'_i \\ t^{-2} c'_i & d_i \end{pmatrix}$$

for all $i > 1$. We saw that $a_i = a'_i$ and $d_i = d'_i$ for all $i > 1$, so

$$h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} a'_i & t^2 b'_i \\ t^{-2} c'_i & d'_i \end{pmatrix} = s h'_i s^{-1}$$

for $s \in SL(2, \mathbb{R})$ given by $s : z \mapsto t^2 z$. Thus there exists $s \in SL(2, \mathbb{R})$ which, for all i, conjugates h_i to h'_i. Hence G and G' are the same group up to conjugation in $SL(2, \mathbb{R})$ or a reflection. $\square$

Note that we can easily extend this result to the case of a group G which is the fundamental group of an admissible Riemann orbisurface of genus $g \geq 1$. We know that any such group contains a hyperbolic element; without loss of generality, label this element h_1. Then the calculations which show that equation (6.11) holds are still valid, as are the calculations which show that $b_i c_j = b'_i c'_j$ for $i \neq j$. An elliptic element R in $SL(2, \mathbb{R})$ is conjugate to an element of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $0 \leq \theta < 2\pi$; a straightforward but messy calculation shows that the off-diagonal entries of R are zero only if $\theta = 0, \pi$. But $\theta = 0$ is the identity element (hence not a generator of our group), and $\theta = \pi$ is excluded. So the remainder of the argument holds in the desired setting.

To prove Theorem 6.5.1, we will need the following result of Stanhope ([32]) which gives an upper bound on the diameter of an orbifold:

Proposition 6.5.4. *Let O be a compact Riemannian orbifold with Ricci curvature bounded below by $\kappa(n-1), \kappa$ real. Fix an arbitrary constant r greater than zero. Then the number of disjoint balls of radius r that can be placed in O is bounded above by a* *number that depends only on κ and the number of eigenvalues of O less than or equal to $\lambda^n_\kappa(r)$.*

In particular the diameter of O is bounded above by a number that depends only on Spec(O) and κ.

So isospectral families of orbifolds with a common uniform lower Ricci curvature bound also have a common upper diameter bound. We are now prepared to prove Theorem 6.5.1.

Proof. Let C be the class of compact orientable hyperbolic orbifolds with cone points of order three and higher, and let S denote the subclass of C containing those orbifolds which are isospectral to O. Note that any member of S is an admissible Riemann orbisurface. We want to show that S is a finite set. We know that $O = \Gamma\backslash\mathbb{H}$ is determined by its fundamental group Γ. Let P be a Dirichlet polygon for Γ. Theorem 5.2.1 tells us that the side-pairing elements of P generate Γ. By Proposition 6.5.3, specifying Γ (up to a reflection or conjugation) is the same as specifying the single, double and triple traces of a set of generators. So we need to show that there are only finitely many possibilities for the single, double and triple traces of the side-pairing elements of P.

First, note that there can be only finitely many isotropy types for the points in an orbisurface in S by Theorem 6.4.8. Also, by Theorem 6.4.9, there are only finitely many cone points in the collection of orbisurfaces in S. So there are only finitely many choices for the trace of any elliptic element in Γ; this implies that if a product of generators of Γ is elliptic, there are only finitely many choices for the trace of such a product. It thus suffices to bound the trace of a hyperbolic element which arises as a product of one, two, or three generators of Γ.

Theorem 6.4.10 tells us that the Laplace spectrum determines (up to finitely many possibilities) the length $\ell(Q)$ of a shortest closed path in the free homotopy class associated to a given hyperbolic conjugacy class Q in Γ. Thus the relation between the trace of Q and $\ell(Q)$ is given by:

$$tr(Q) = \pm 2 \cosh \frac{1}{2}\ell(Q).$$

Note that $\frac{1}{2}\ell(Q)$ is bounded by D = diameter of O. Thus the single traces of the hyperbolic conjugacy classes are bounded by $2\cosh D$. We fix a point $p \in P$ and determine an upper bound for $\operatorname{dist}(p, g_2 \circ g_1(p))$, where g_1 and g_2 are side-pairing elements of P and $g_2 \circ g_1$ is hyperbolic. We have

$$\operatorname{dist}(p, g_2 \circ g_1(p)) \leq \operatorname{dist}(p, g_1(p)) + \operatorname{dist}(g_1(p), g_2 \circ g_1(p))$$

and each term on the right side is bounded by $2D$. Thus

$$tr(g_2 \circ g_1) \leq 2 \cosh 2D.$$

A similar argument shows that the trace of the product of three side-pairing elements which is hyperbolic is bounded by $2\cosh 3D$.

By Proposition 6.5.4, there is a common upper bound on the diameter of any orbifold in S. So there are only finitely many possibilities for the single, double, and triple traces of a hyperbolic element which arises as a product of side-pairing elements of P, a Dirichlet polygon for Γ.

Thus we have shown that there are only finitely many ways to choose the generators of Γ, and hence to choose an element in S. $\square$

Remark 6.5.5. *Buser [6] has an explicit bound of* e^{720g^2} *on the cardinality of a set of* *pairwise non-isometric isospectral compact Riemann surfaces of genus* $g \geq 2$*. There are examples in which the cardinality of such a set grows faster than polynomially in* g *(see [4]). Many of the tools developed in this work, especially the material in Chapters 2 and 3, are intended to be the first steps in the search for such examples and for an explicit bound in the case of compact Riemann orbisurfaces.*

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