Inverse Spectral Problems on Riemannian Orbifolds

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What is an orbifold?

Examples

1. Let Γ be a group acting properly discontinuously on a manifold M with fixed point set of codimension 2 or greater. Then the quotient space M/Γ is an orbifold.

2. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p

Example Orbifolds arising from triangle groups: topologically a 2-sphere, with three cone points

Points in O with nontrivial isotropy groups are called *singular points*, and the collection of all such singular points is the *singular set* Σ_O.

EXAMPLE Manifolds are orbifolds for which the singular set is empty.

Why are orbifolds of interest?

1. Visual way to understand group acting on a space
2. Easiest singular spaces
3. Crystallography
4. String theory
5. Study of 3-manifolds

Riemannian Orbifolds

Construct Riemannian metric on O by defining metrics locally via coordinate charts and patching metrics together using a partition of unity.

Structures must be invariant under local group actions.

Results of local analysis hold, but global results may not hold or take new form.

Every point p in a Riemannian orbifold has a *fundamental coordinate chart*.

Definition Let O be a compact Riemannian orbifold. A map $f : O \to \mathbb{R}$ is a *smooth function* on O if for every coordinate chart $(U, \tilde{U}/\Gamma, \pi)$ on O, the lifted function $\tilde{f} = f \circ \pi$ is a smooth function on $\tilde{U}$.

If O is a compact Riemannian orbifold and f is a smooth function on O, then we define the Laplacian Δf of f by lifting f to local covers. That is, we lift f to $\tilde{f} = f \circ \pi$ via a coordinate chart $(U, \tilde{U}/\Gamma, \pi)$. We denote the Γ-invariant metric on $\tilde{U}$ by g_{ij} and set $\rho = \sqrt{det(g_{ij})}$. Then we can define

$$\Delta \tilde{f} = \frac{1}{\rho} \sum_{i,j=1}^{n} \frac{\partial}{\partial \tilde{x^i}} (g^{ij} \frac{\partial f}{\partial \tilde{x^j}} \rho).$$

THEOREM *(Chiang) Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*

2. *Each eigenvalue λ_i has finite multiplicity.*

3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta \phi_i = \lambda_i \phi_i$.*

1966: Kac asked “Can one hear the shape of a drum?”

Examples: Milnor, Vignéras, Sunada’s method, submersion method

2000: Dianu showed that every indexed one-pointed torus is uniquely determined up to isometry by the first few lengths in its length spectrum

2002: Gordon and Rossetti showed that the middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

2003: Gordon, Greenwald, Webb, Zhu calculated the first few invariants of the heat expansion for bad orbifolds

Definition Let O be a 2-orbifold with r corner reflectors of orders $n_1, \ldots, n_r$ and s cone points of orders $m_1, \ldots, m_s$. Then we define the (orbifold) Euler characteristic of O to be

$$\chi(O) = \chi(X_0) - \frac{1}{2}\sum_{i=1}^{r}(1 - \frac{1}{n_i}) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}),$$

where $\chi(X_O)$ is the Euler characteristic of the underlying space of O.

THEOREM *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O),$$

where K is the curvature and $\chi(O)$ is the orbifold Euler characteristic.

THEOREM *(Farsi) Let O be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ we have*

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O)\frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$. Here $B_0^n(1)$ denotes the n-dimensional unit ball in Euclidean space.

Consequences:

1. Laplace spectrum determines an orbifold's dimension and volume
2. Dimension 2: spectrum determines an orbifold's Euler characteristic

PROPOSITION *Fix* $g \geq 0$ *and* $m \geq 2$. *Let* O *be an orientable hyperbolic 2-orbifold of genus* g *with exactly one cone point of order* m. *Let* O' *be in the class of orientable hyperbolic 2-orbifolds of genus* g *with cone points of orders 2 and higher, and suppose that* O *is isospectral to* O'. *Then* O' *must have exactly one cone point, and its order is also* m.

Proof Let O and O' be orientable hyperbolic 2-orbifolds with the same genus , i.e. $\chi(X_O) = \chi(X_{O'})$. Further suppose that O is isospectral to O'. Then $\chi(O) = \chi(O')$. Suppose that O' has one cone point of order n_1. It follows that

$$\frac{1}{m} = \frac{1}{n_1},$$

or $m = n_1$.

Now suppose that O' has two cone points of orders n_1 and n_2. Then

$$\frac{1}{m} + 1 = \frac{1}{n_1} + \frac{1}{n_2}.$$

But $n_i \geq 2$ for $i = 1, 2$, so $\frac{1}{n_1} + \frac{1}{n_2} \leq 1$. This is a contradiction, hence O and O' are not isospectral. This argument is easily extended to $k > 2$ cone points of orders $n_1, \ldots, n_k$. $\square$

We can extend this proposition to the case of two orbifolds with different underlying spaces.

PROPOSITION *Let O be an orientable hyperbolic 2-orbifold of genus $g_0 \geq 0$ with k cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable hyperbolic 2-orbifold of genus $g_1 \geq g_0$ with l cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k + h)$, then O is not isospectral to O'.*

COROLLARY *Fix $g \geq 0$. Let O be an orientable hyperbolic 2-orbifold of genus g with k cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable hyperbolic 2-orbifold of genus g with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then O is not isospectral to O'.*

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of orbifold Riemann surfaces. Specifically, we show

THEOREM *Let O be a compact hyperbolic orientable 2-orbifold with genus $g \geq 1$ and cone points of order three and higher. Then in the class of compact hyperbolic orientable orbifolds, there are only finitely many members which are isospectral to O.*

Future Directions:

1. Explicit bounds on the size of isospectral sets
2. Examples of large families
3. Understand orbifold injectivity radius
4. What properties of orbifolds are spectrally determined?