Can one hear the shape of a ...?

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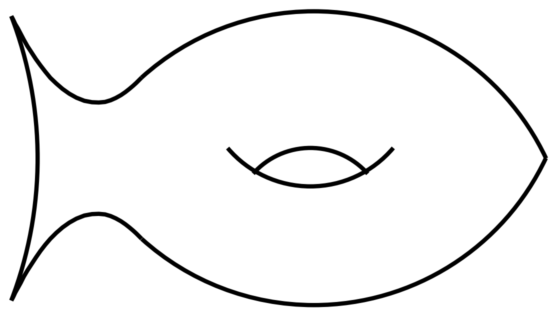

Scripps College Mathematics Department Colloquium

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Slides available from `http://www.math.mcgill.ca/~dryden/`

The Plan:

1. Historical Motivation

2. Vibrating Strings

3. Drums and Manifolds

4. Orbifolds

Motivation from Chemistry

Observed vibration frequencies (spectrum) of system in lab

Applied information to astronomical observations to identify molecules in space

QUESTION *How do structure of system and vibrational frequencies of system relate?*

Development of quantum mechanics to provide theoretical foundation for spectroscopy

A wise man once said...

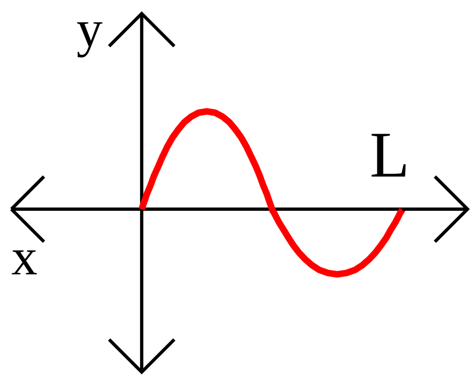
Sir Arthur Schuster, 1882:

We know a great deal more about the forces which produce the vibrations of sound than about those which produce the vibrations of light. To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction.

Vibrating Strings

Setup: string of length L with uniform density and tension, fixed endpoints

Pluck the string:



Describe motion of string with function $f(x, t)$, which gives vertical displacement

Constraints on $f(x, t)$

Boundary conditions:

- $f(0, t) = 0$
- $f(L, t) = 0$

Wave equation:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

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- $f(L,t) = 0$

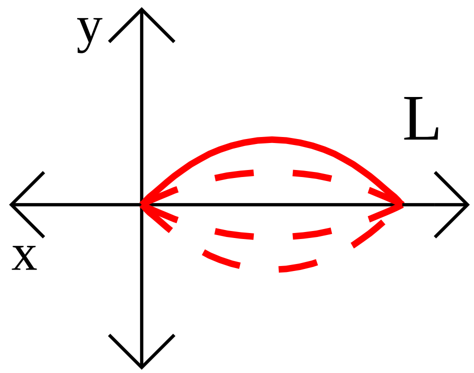
Wave equation:

$$\underbrace{\frac{\partial^2 f}{\partial t^2}}_{\text{acceleration}} = \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{curvature}}$$

Solving the Wave Equation

Look for stationary solutions, i.e. solutions $f(x,t)$ such that

$$f(x,t) = g(x)h(t)$$



$g(x)$ gives shape

$h(t)$ measures amplitude

Plugging in our stationary solution

Substitute solution $f(x,t) = g(x)h(t)$ into wave equation

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or

$$\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)}$$

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Solving two equations

Rewrite $\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)} = -\lambda$ as

1. $g''(x) = -\lambda g(x)$

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General solutions are

1. $g(x) = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$
2.

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1. $g(x) = A \sin \sqrt{\lambda} x$ $\qquad\qquad$ $\sqrt{\lambda} L = n\pi$
2. $h(t) = C \sin \sqrt{\lambda} t + D \cos \sqrt{\lambda} t$

Restrictions on λ

We have $\sqrt{\lambda}L = n\pi$ from boundary conditions on $g(x)$

Frequency of oscillation given by $h(t)$ is $\frac{\sqrt{\lambda}}{2\pi}$

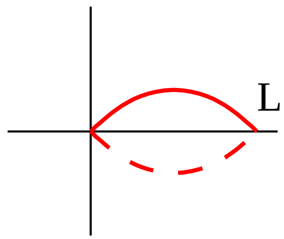
Thus

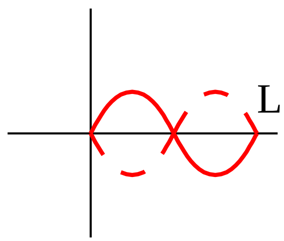
$$\text{frequency} = \frac{\sqrt{\lambda}}{2\pi} = \frac{n}{2L},$$

and the string is allowed to vibrate at frequencies $\frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \ldots$

Waveforms

Specific waveforms oscillate at specific frequencies $\frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \ldots$



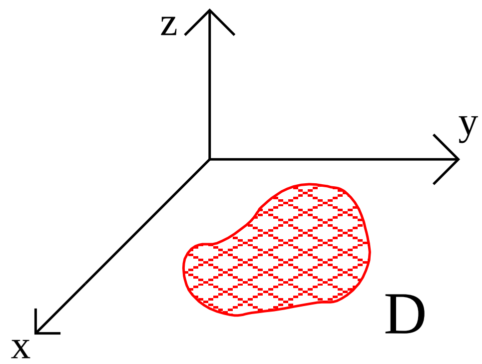


Waveforms form basis for vector space of motion functions $f(x,t)$

Can "hear" the shape (length) of a string!

Can one hear the shape of a drum?

D = compact domain in Euclidean plane



Describe motion with function $f(x, y, t)$

Constraints on $f(x, y, t)$:

- $f(x_0, y_0, t) = 0$ for all (x_0, y_0) on boundary of D
- $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} := \Delta f$

Waveforms Again

Sound of drum given by list of frequencies associated to waveforms

$$f(x, y, t) = g(x, y)h(t)$$

Substitute solution into wave equation:

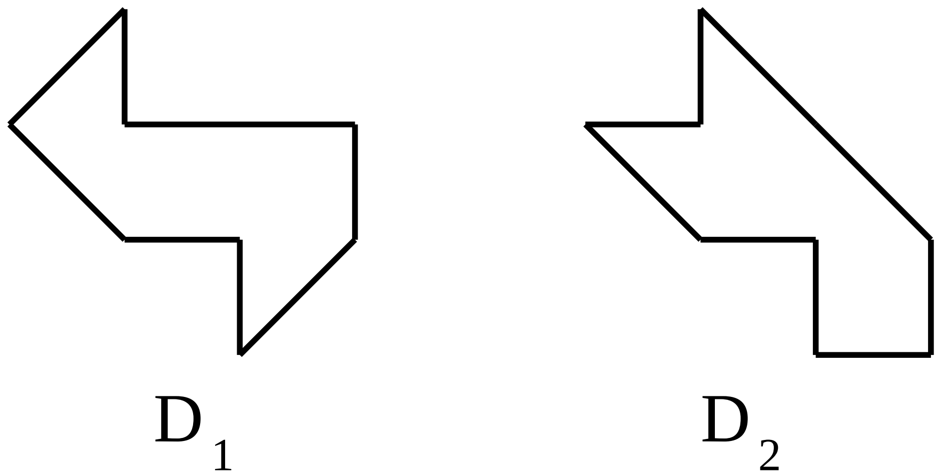
$$\frac{\Delta g}{g} = \frac{h''}{h} = -\lambda$$

Frequencies of vibration = Eigenvalues of Δ on D

Cannot explicitly calculate list of frequencies in general

Can hear area and perimeter of drumhead

You cannot hear the shape of a drum!



Manifolds

M = compact Riemannian manifold

$\Delta = -div\ grad$

Big Question *How much geometric information about M is encoded in the eigenvalue spectrum of Δ?*

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Answers:

- dimension
- volume
- M = surface: Euler characteristic, hence genus

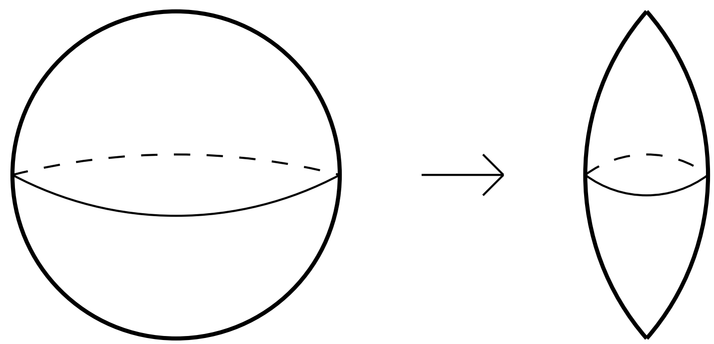
What is an orbifold?

Examples

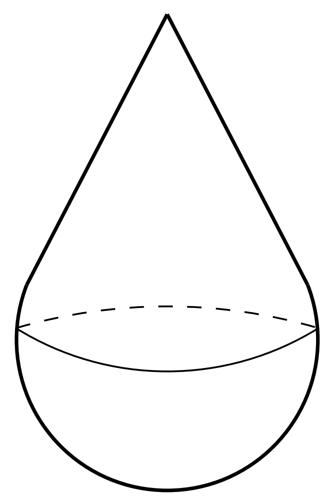
1. Manifolds

2. M/Γ, where Γ is a group acting "nicely" on a manifold M

Let $M = S^2$, and let Γ be the group of rotations of order 3 about the north-south axis. Then M/Γ is a $(3,3)$-football.



3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Riemannian Orbifolds

Construction of Riemannian metric on O:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\mathrm{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$
- Each eigenvalue λ_i has finite multiplicity.

Inverse Spectral Geometry of Orbifolds

- Gordon, Webb and Wolpert (1992): used orbifolds in construction of drum examples
- Gordon and Rossetti (2003): middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds
- Gordon, Greenwald, Webb and Zhu (2003): spectral invariant for footballs and teardrops
- Shams, Stanhope and Webb: there exist arbitrarily large (but always finite) isospectral sets, where each element in a given set has points of distinct isotropy

Listening to Orbifolds

O = compact Riemannian orbifold

$\Delta = -\mathit{div}\ \mathit{grad}$ (locally)

BIG QUESTION *How much geometric information about O is encoded in the eigenvalue spectrum of Δ?*

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Answers:

- dimension
- volume
- orbisurfaces: genus???
- isospectral nonisometric Riemann orbisurfaces???

Tools in Dimension 2

O: orbisurface with s cone points of orders $m_1, \ldots, m_s$

Define the (orbifold) Euler characteristic of O to be

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}).$$

THEOREM *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O).$$

Euler characteristic is spectrally determined, but unknown if spectrum determines genus

Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of Riemann orbisurfaces. Specifically, we show

THEOREM *(D.) Let O be a compact Riemann orbisurface with genus $g \geq 1$. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Sounds and Lengths

THEOREM *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

length spectrum: sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order

THEOREM *(D.-Strohmaier) If two compact Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra and a sum involving the orders of the cone points. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Explicit Bounds

THEOREM *(Buser) Let S be a compact Riemann surface of genus $g \geq 2$. At most e^{720g^2} pairwise non-isometric compact Riemann surfaces are isospectral to S.*

No g-independent upper bound is possible

Brooks, Gornet, and Gustafson examples: cardinality of set grows faster than polynomially in g

Bounds for Riemann Orbisurfaces

- Cubic pseudographs (D.)
- Fenchel-Nielsen parameters (D.)
- Collar theorem (D.-Parlier)
- Bers' theorem (D.-Parlier)
- Understanding of geodesic behavior

Future Directions

- How do geodesics on orbifolds behave?
- For what classes of orbifolds are the isotropy types spectrally determined?
- What is the relationship between the spectrum of a Riemann orbisurface and that of the Riemann surface which finitely covers it?