Hearing the weights of weighted projective planes

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The Plan

1. Introduction and main result
2. Weighted projective spaces
3. Heat invariants
4. Kähler metrics
5. Proof of main result

Manifolds

M = compact Riemannian manifold

Big Question Which properties of M are determined by the spectrum of the Laplacian acting on functions, or more generally, on differential forms?

Answers:

- dimension
- volume
- M = surface:
  Euler characteristic, hence genus
- isospectral nonisometric planar domains
- complex projective spaces, Einstein manifolds, Kähler manifolds

Orbifolds

Question Which orbifold properties of a Riemannian orbifold are determined by the spectrum of its Laplacian?

Answers:

- (Farsi) dimension and volume
- (Stanhope) finitely many isotropy types in isospectral set with Ricci curvature bounded below
- (DGGW) weights of a football, $\mathbb{C}P^1(N_1, N_2)$, with any Riemannian orbifold metric

MAIN RESULT Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities, equipped with any Kähler orbifold metric. Then the spectra of its Laplacian acting on functions and 1-forms determine the weights N_1, N_2 and N_3.

Weighted projective spaces

Let $\mathbf{N} = (N_1, \ldots, N_{m+1})$ be a vector of positive integers which are pairwise relatively prime. The weighted projective space

$$\mathbb{C}P^m(\mathbf{N}) := \mathbb{C}P^m(N_1, \ldots, N_{m+1}) := (\mathbb{C}^{m+1})^*/\sim,$$

where

$$((z_1, \ldots, z_{m+1}) \sim (\lambda^{N_1} z_1, \ldots, \lambda^{N_{m+1}} z_{m+1}),\ \lambda \in \mathbb{C}^*),$$

is a compact orbifold. It has $m+1$ isolated singularities at the points $[1 : 0 : \cdots : 0]$, $\ldots$, $[0 : \cdots : 0 : 1]$, with isotropy groups $\mathbb{Z}_{N_1}$, $\ldots$, $\mathbb{Z}_{N_{m+1}}$.

Note that $\mathbb{C}P^m(\mathbf{1})$ is the usual smooth projective space $\mathbb{C}P^m$.

Heat invariants

THEOREM *(Donnelly, DGGW) Let O be a Riemannian orbifold with **isolated** singularities and let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to*

$$I_0 + I_{singular} \tag{1}$$

where I_0 is the "smooth" part, i.e.

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k$$

and $a_k(O)$ are the usual heat invariants. The $I_{singular}$ part is the sum of infinitesimal local contributions from each singular point.

Heat invariants for weighted projective planes

$O = \mathbb{C}P^2(N_1, N_2, N_3)$ is a weighted projective plane

N_1, N_2, N_3 pairwise relatively prime

Then the first few terms in the asymptotic expansion are:

- degree -2 term: $a_0 = vol(O)$
- degree -1 term: $a_1 = \frac{1}{6}\int_O \tau dvol_O(g)$
- degree 0 term: $\frac{a_2}{16\pi^2} + b_0$, where

  $$a_2(O) = \frac{1}{360}\int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$$

  and b_0 involves N_1, N_2, N_3.

That b_0 term

More specifically,

$$
\begin{aligned}
b_0 \quad = \quad & \frac{1}{N_1} \sum_{j=1}^{N_1-1} \frac{1}{16 \sin^2(\frac{N_2 j\pi}{N_1}) \sin^2(\frac{N_3 j\pi}{N_1})} \\
+ \quad & \frac{1}{N_2} \sum_{j=1}^{N_2-1} \frac{1}{16 \sin^2(\frac{N_1 j\pi}{N_2}) \sin^2(\frac{N_3 j\pi}{N_2})} \\
+ \quad & \frac{1}{N_3} \sum_{j=1}^{N_3-1} \frac{1}{16 \sin^2(\frac{N_1 j\pi}{N_3}) \sin^2(\frac{N_2 j\pi}{N_3})}.
\end{aligned}
$$

No closed formula for b_0 in terms of N_1, N_2, N_3!

Conjecture The spectrum of the Laplacian acting on functions determines N_1, N_2, N_3.

Kähler metrics

Let (O^{2m}, J, ω) be a Kähler orbifold, with Kähler metric $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$. Denote by c_1 and c_2 its first and second Chern classes. Then

$$\mathrm{vol}(O) = \int_O 1 \ dvol_O(g) = \frac{1}{m!}\int_O \omega^m$$

and

$$\int_O \tau \ dvol_O(g) = \frac{4\pi}{(m-1)!}\int_O c_1 \wedge \omega^{m-1}.$$

When $2m = 4$, we can write

$$\begin{aligned} \frac{a_2}{16\pi^2} &= \frac{1}{90}\int_O c_2 - \frac{1}{120}\int_O c_1 \wedge c_1 \\ &+ \frac{1}{60\cdot 16\pi^2}\int_O \tau^2 \ dvol_O(g) \end{aligned}$$

Localization in equivariant cohomology

$O = \mathbb{C}P^2(N_1, N_2, N_3)$ is a weighted projective plane

N_1, N_2, N_3 pairwise relatively prime

Use Hamiltonian action of $S^1 \times S^1$, together with localization in equivariant cohomology, to compute:

$$\int_O c_1 \wedge \omega = \frac{N_1 + N_2 + N_3}{\sqrt{N_1 N_2 N_3}} \sqrt{2\mathrm{vol}(O)}$$

$$\int_O c_1 \wedge c_1 = \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3}$$

$$\int_O c_2 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}$$

Sketch of proof of main result

$O = \mathbb{C}P^2(N_1, N_2, N_3)$

N_1, N_2, N_3 pairwise relatively prime

g any orbifold Kähler metric

The spectrum of the Laplacian acting on functions determines

$$a_0 = \mathrm{vol}(O)$$

and

$$\begin{aligned} a_1 &= \frac{1}{6}\int_O \tau \; dvol_O(g) = \frac{2\pi}{3}\int_O c_1 \wedge \omega \\ &= \frac{2\pi}{3}\frac{N_1 + N_2 + N_3}{\sqrt{N_1 N_2 N_3}}\sqrt{2\mathrm{vol}(O)}. \end{aligned}$$

Hence we can hear

$$b := \frac{(N_1 + N_2 + N_3)^2}{N_1 N_2 N_3} = \int_O c_1 \wedge c_1.$$

Sketch of proof continued

Spectrum of Laplacian on functions determines

$$\begin{aligned} T_0 = \frac{a_2}{16\pi^2} + b_0 &= \frac{1}{90}\int_O c_2 - \frac{1}{120}\int_O c_1 \wedge c_1 \\ &+ \frac{1}{60 \cdot 16\pi^2}\int_O \tau^2 dvol_O(g) + b_0. \end{aligned}$$

Spectrum of Laplacian on 1-forms determines

$$\begin{aligned} T_1 &= -\frac{11}{90}\int_O c_2 - \frac{7}{60}\int_O c_1 \wedge c_1 \\ &+ 4\left(\frac{1}{60 \cdot 16\pi^2}\int_O \tau^2 dvol_O(g) + b_0\right). \end{aligned}$$

Together these spectra determine

$$12(T_1 - 4T_0) = -\int_O c_1 \wedge c_1 - 2\int_O c_2$$

so we can hear $c := \int_O c_2 = \frac{N_1 N_2 + N_1 N_3 + N_2 N_3}{N_1 N_2 N_3}$ and hence also

$$\int_O c_1 \wedge c_1 - 2\int_O c_2 = \frac{N_1^2 + N_2^2 + N_3^2}{N_1 N_2 N_3} = d.$$

One final whack...

LEMMA *The numbers b, c, d defined on the previous slides determine N_1, N_2, N_3 uniquely up to permutation.*

REMARK Under certain additional assumptions, we can determine the weights using only the function spectrum.

THEOREM *Let $O = \mathbb{C}P^2(N_1, N_2, N_3)$ where the weights N_1, N_2, N_3 are known. Then the spectrum on functions determines whether O is endowed with the extremal metric.*