Bounding the Size of Isospectral Sets of Orbifolds

Emily Dryden
Dartmouth College

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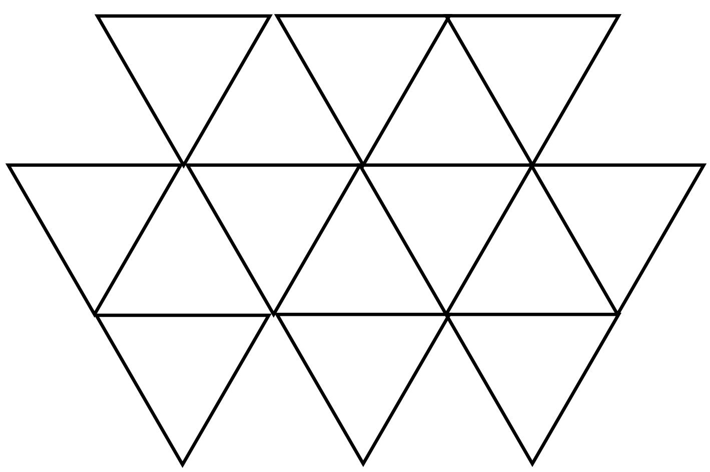
`http://www.math.dartmouth.edu/~edryden`

What is an orbifold?

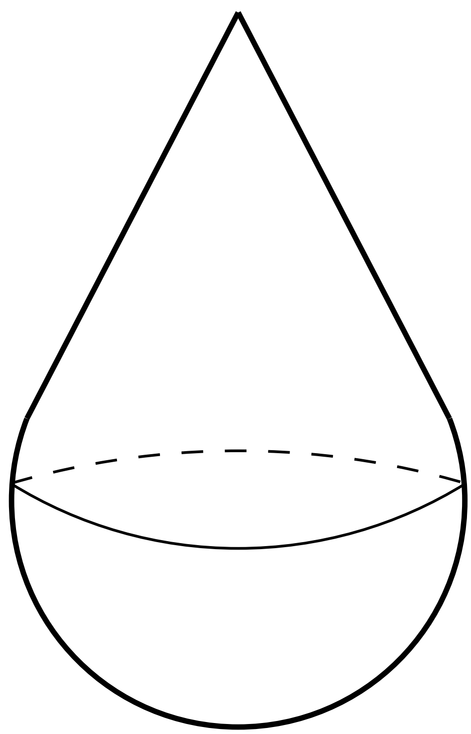
Examples

- Manifolds
- M/Γ, where Γ is a group acting properly discontinuously on a manifold M

properly discontinuously: for any compact subset C of M, $\{g \in \Gamma \mid gC \cap C \neq \emptyset\}$ is finite



- $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Why are orbifolds of interest?

1. Visual way to understand group acting on a space
2. Easiest singular spaces
3. Crystallography
4. String theory
5. Study of 3-manifolds

Riemann Orbisurfaces

orbisurface: two-dimensional orbifold

Riemann: Need a metric! Construction of Riemannian metric on O analogous to manifold case

Assume orientable

Riemannian Orbifolds

Structures must be invariant under local group actions.

(Every compact Riemann orbisurface is finitely covered by a compact Riemann surface.)

Use local definitions (e.g. function, Laplacian)

Results of local analysis hold, but global results may not hold or take new form.

Definition Let O be a compact Riemannian orbifold. A map $f : O \to \mathbb{R}$ is a *smooth function* on O if for every coordinate chart $(U, \tilde{U}/\Gamma, \pi)$ on O, the lifted function $\tilde{f} = f \circ \pi$ is a smooth function on $\tilde{U}$.

If O is a compact Riemannian orbifold and f is a smooth function on O, then we define the Laplacian Δf of f by lifting f to local covers. That is, we lift f to $\tilde{f} = f \circ \pi$ via a coordinate chart $(U, \tilde{U}/\Gamma, \pi)$. We denote the Γ-invariant metric on $\tilde{U}$ by g_{ij} and set $\rho = \sqrt{det(g_{ij})}$. Then we can define

$$\Delta \tilde{f} = \frac{1}{\rho} \sum_{i,j=1}^{n} \frac{\partial}{\partial \tilde{x^i}} (g^{ij} \frac{\partial f}{\partial \tilde{x^j}} \rho).$$

Theorem (Chiang) *Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*

2. *Each eigenvalue λ_i has finite multiplicity.*

3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta\phi_i = \lambda_i \phi_i$.*

The Heat Equation

$$\Delta F = -\frac{\partial F}{\partial t}$$

where $F(x,t)$ is heat at a point x at time t

With initial data $f : O \to \mathbf{R}$, $F(x,0) = f(x)$, solution of heat equation given by

$$F(x) = \int_O K(x,y,t) f(y) dy$$

What is K?

$K : O \times O \times \mathbf{R}_+^* \to \mathbf{R}$, C^∞ function given by the convergent series

$$K(x, y, t) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

K is the *fundamental solution of the heat equation* on O, or the *heat kernel* on O.

Physical interpretation: $K(x, y, t)$ is the temperature at time t at the point y when a unit of heat is placed at the point x

Weil's Asymptotic Formula

THEOREM *(Farsi) Let O be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ we have*

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O)\frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$.

The Laplace spectrum determines an orbifold's volume and dimension.

Tools in Dimension 2

Define the (orbifold) Euler characteristic of O, a compact orbisurface with cone points of orders $m_1, \ldots, m_s$, to be

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}).$$

THEOREM *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O).$$

Obstructions to Isospectrality

Proposition *Let O be an orientable hyperbolic 2-orbifold of genus $g_0 \geq 0$ with k cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable hyperbolic 2-orbifold of genus $g_1 \geq g_0$ with l cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k + h)$, then O is not isospectral to O'.*

Corollary *Fix $g \geq 0$. Let O be an orientable 2-orbifold of genus g with k cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable 2-orbifold of genus g with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then O is not isospectral to O'.*

Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of orbifold Riemann surfaces. Specifically, we show

THEOREM *Let O be a compact orientable hyperbolic 2-orbifold with genus $g \geq 1$ and cone points of order three and higher. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Huber's Theorem for Compact Riemann Surfaces

THEOREM *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

length spectrum: sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order

Idea of Proof of Huber's Theorem

Fundamental domain argument leads to length trace formula

Use known eigenfunction expansion of heat kernel

$$K(x,y,t) = \sum_i e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

Plug heat kernel into length trace formula to get Selberg Trace Formula

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi}\int_{-\infty}^{\infty} r h(r)\tanh(\pi r)dr$$

$$+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)]$$

Selberg Trace Formula for Compact Riemann Orbisurfaces

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr$$

$$+ \sum_{\substack{\{R\} \\ \text{elliptic}}} \frac{1}{2m(R) \sin\theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1 + e^{-2\pi r}} h(r) dr$$

$$+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)]$$

A Partial Analog of Huber's Theorem

THEOREM *If two compact orientable hyperbolic 2-orbifolds are Laplace isospectral, then we can determine their length spectra up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Sketch of Proof (Laplace spectrum determines length spectrum):

Use appropriate version of Selberg Trace Formula

- Know volume from Weil's asymptotic formula
- Determine elliptic summand up to finitely many possibilities
- Read off lengths

Elliptic Summand

THEOREM *(Stanhope) Given a collection of isospectral orientable compact Riemannian orbifolds that share a uniform lower bound on Ricci curvature, there are only finitely many possible isotropy types, up to isomorphism, for points in an orbifold in the collection.*

THEOREM *(Stanhope) Given a collection of isospectral compact Riemannian orbifolds with only isolated singularities that share a uniform lower bound on sectional curvature, there is an upper bound on the number of singular points in any orbifold in the collection.*

Reading Off the Lengths

Up to finitely many possibilities, we know the function

$$f(t) = \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} e^{-(\ln N(P))^2/4t}$$

Consider $b(t) = f(t)e^{\omega^2/4t}$.

Take the limit of $b(t)$ as $t \downarrow 0$.

Unique $\omega > 0$ for which limit is finite and nonzero

$\omega = \ell(\gamma_1)$, where γ_1 is shortest primitive closed geodesic in O

A Complete Analog of Huber's Theorem?

Consider the wave equation

$$\Delta f = -\frac{\partial^2 f}{\partial t^2}$$

Can study the wave trace

$$\sum_j e^{-it\sqrt{\lambda_j}}$$

Poisson relation (Stanhope and Uribe): singularities of wave trace contained in set of lengths of closed geodesics

Finiteness of Isospectral Sets

THEOREM *Let O be a compact orientable hyperbolic 2-orbifold with genus $g \geq 1$ and cone points of order three and higher. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Sketch of Finiteness Proof:

$O = \mathbb{H}/\Gamma$

- O determined by Γ
- P: Dirichlet polygon for Γ
- Side-pairing elements of P generate Γ
- Specify Γ via traces of set of generators
- Bound traces...

Bounding the Traces

Q: hyperbolic conjugacy class

E: elliptic conjugacy class

- $tr(Q) = \pm 2 \cosh \frac{1}{2}\ell(Q) \leq \pm 2 \cosh D$

- $tr(E) = 2 \cos \theta < 2 \cosh D$

- double, triple traces bounded by function of D

- bound on D from Stanhope

Explicit Bounds

THEOREM *(Buser) Let S be a compact Riemann surface of genus $g \geq 2$. At most e^{720g^2} pairwise non-isometric compact Riemann surfaces are isospectral to S.*

No g-independent upper bound is possible

Brooks, Gornet, and Gustafson examples: cardinality of set grows faster than polynomially in g

Ingredients

- Fenchel-Nielsen parameters
- Collar theorem
- Bers' theorem

Future Directions

1. Explicit bounds on the size of isospectral sets
2. Examples of large families
3. Understand orbifold injectivity radius
4. What properties of orbifolds are spectrally determined?