Isospectrality of Two-Dimensional Riemannian Orbifolds

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What is an orbifold?

Examples

1. Manifolds

2. M/Γ, where Γ is a group acting properly discontinuously on a manifold M

3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p

Riemannian Orbifolds

Construction of Riemannian metric on O analogous to manifold case

Structures must be invariant under local group actions.

Use local definitions (e.g. function, Laplacian)

Results of local analysis hold, but global results may not hold or take new form.

THEOREM *(Chiang) Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*
2. *Each eigenvalue λ_i has finite multiplicity.*
3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta\phi_i = \lambda_i\phi_i$.*

Tools in Dimension 2

O: 2-orbifold with r corner reflectors of orders $n_1, \ldots, n_r$ and s cone points of orders $m_1, \ldots, m_s$

Define the (orbifold) Euler characteristic of O to be

$$\chi(O) = \chi(X_0) - \frac{1}{2}\sum_{i=1}^{r}(1 - \frac{1}{n_i}) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}).$$

Theorem *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O).$$

THEOREM *(Farsi) Let O be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ we have*

$$N(\lambda) \sim (\mathrm{Vol}\ B_0^n(1))(\mathrm{Vol}\ O)\frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$.

Consequences:

1. Laplace spectrum determines an orbifold's dimension and volume
2. Dimension 2: spectrum determines an orbifold's Euler characteristic

Obstructions to Isospectrality

PROPOSITION *Let O be an orientable hyperbolic 2-orbifold of genus $g_0 \geq 0$ with k cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable hyperbolic 2-orbifold of genus $g_1 \geq g_0$ with l cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k + h)$, then O is not isospectral to O'.*

COROLLARY *Fix $g \geq 0$. Let O be an orientable 2-orbifold of genus g with k cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be an orientable 2-orbifold of genus g with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then O is not isospectral to O'.*

Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of orbifold Riemann surfaces. Specifically, we show

Theorem *Let O be a compact orientable hyperbolic 2-orbifold with genus $g \geq 1$ and cone points of order three and higher. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

A Partial Analog of Huber's Theorem

THEOREM *If two compact orientable hyperbolic 2-orbifolds are Laplace isospectral, then we can determine their length spectra up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Sketch of Proof:

Use appropriate version of Selberg Trace Formula

- Know volume from Weil's asymptotic formula
- Know only finitely many possible isotropy types, up to isomorphism, by theorem of Stanhope
- Know only finitely many cone points by theorem of Stanhope
- Can determine sum over elliptic classes up to finitely many possibilities
- Read off lengths in usual way

Sketch of Finiteness Proof:

$O = \mathbb{H}/\Gamma$

- O determined by Γ
- P: Dirichlet polygon for Γ
- Side-pairing elements of P generate Γ
- Specify Γ via traces of set of generators
- Bound traces...

Bounding the Traces

Q: hyperbolic conjugacy class

E: elliptic conjugacy class

- $tr(Q) = \pm 2\cosh\frac{1}{2}\ell(Q) \leq \pm 2\cosh D$
- $tr(E) = 2\cos\theta < 2\cosh D$
- double, triple traces bounded by function of D
- bound on D from Stanhope

Future Directions

1. Explicit bounds on the size of isospectral sets
2. Examples of large families
3. Understand orbifold injectivity radius
4. What properties of orbifolds are spectrally determined?