Using heat invariants to hear the geometry of orbifolds

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The Plan

1. Historical motivation
2. Orbifolds
3. Heat kernel and heat invariants
4. Applications

Historical Motivation

Chemistry: identify elements by spectral "fingerprints"

Physics: development of quantum mechanics

Mathematics: how are knowledge of structure and knowledge of spectrum related?

Manifolds

M = compact Riemannian manifold
$\Delta = -div\ grad$

Big Question How much geometric information about M is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- M = surface:
  Euler characteristic, hence genus

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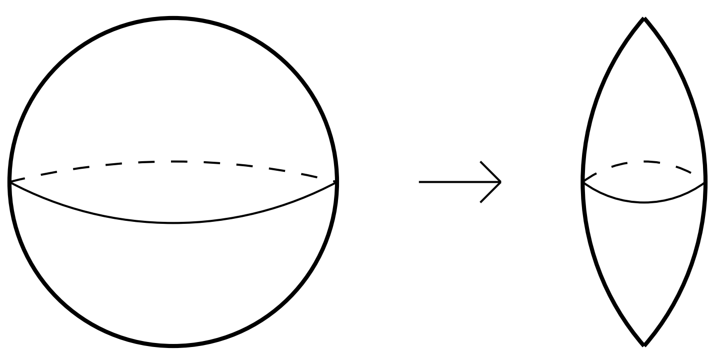
- M = surface:
  Euler characteristic, hence genus
- round spheres characterized by spectra
- isospectral nonisometric Riemann surfaces
- isospectral nonisometric planar domains

What is an orbifold?

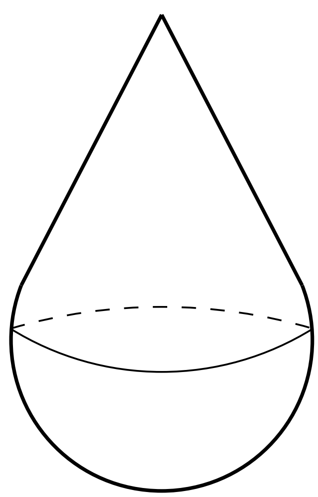
Examples

1. Manifolds
2. M/Γ, where Γ is a group acting "nicely" on a manifold M

Let $M = S^2$, and let Γ be the group of rotations of order 3 about the north-south axis. Then M/Γ is a $(3,3)$-football.



3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Riemannian Orbifolds

Construction of Riemannian metric on O:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\mathrm{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$
- Each eigenvalue λ_i has finite multiplicity.
- Orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions

Listening to Orbifolds

O = compact Riemannian orbifold
$\Delta = -\mathit{div}\ \mathit{grad}$ (locally)

Big Question How much topological or geometric information about O is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- orbisurfaces: genus???
- isotropy type???

Some results

- Gordon and Rossetti (2003): middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

- Shams, Stanhope and Webb (2005): there exist arbitrarily large (but always finite) isospectral sets, where each element in a given set has points of distinct isotropy

- D-Strohmaier (2005): for compact orientable surfaces with singularities and a metric of constant curvature -1, the spectrum determines the isotropy types

Heating things up

Heat operator L on $\mathcal{O}$ defined by $L = \Delta + \partial/\partial t$

Heat equation: $Lu = 0$

Definition We say that $K : (0, \infty) \times \mathcal{O} \times \mathcal{O} \to \mathbf{R}$ is a *fundamental solution* of the heat equation, or *heat kernel*, if it satisfies:

1. K is C^0 in the three variables, C^1 in the first, and C^2 in the second;
2. $(\frac{\partial}{\partial t} + \Delta_x)K(t, x, y) = 0$ where Δ_x is the Laplacian with respect to the second variable;
3. $\lim_t \to 0^+ K(t, x, \cdot) = \delta_x$ for all $x \in \mathcal{O}$.

The heat kernel

Proposition *If the heat kernel exists, then it is unique and is given by* $\sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$.

Showing existence:

1. Construct a parametrix
2. Follow standard construction of heat kernel from parametrix (e.g. Berger-Gauduchon-Mazet)
3. Construction uses local structure of orbifold as quotient of manifold by finite group actions

Get asymptotic expansion of trace of heat kernel as $t \to 0^+$

Asymptotic expansion

THEOREM *Let O be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to*

$$I_0 + \sum_{N \in S(O)} \frac{I_N}{|Ist(N)|} \tag{1}$$

where $S(O)$ is the set of C-strata of O. This asymptotic expansion is of the form

$$(4\pi t)^{-dim(O)/2} \sum_{j=0}^{\infty} c_j t^{\frac{j}{2}}. \tag{2}$$

Huh?!?

I_0 is the "smooth" part, i.e.

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k$$

$a_k(O)$ are the usual heat invariants, e.g.

- $a_0(O) = vol(O)$
- $a_1(O) = \frac{1}{6} \int_O \tau(x) dvol_O(x)$
- If O is finitely covered by a Riemannian manifold M, say $O = G \backslash M$, then

$$a_k(O) = \frac{1}{|G|} a_k(M).$$

I_N is the "singular" part:

$$I_N = \sum_{\gamma \in Ist^*(N)} I_{N,\gamma}$$

where

$$I_{N,\gamma} := (4\pi t)^{-dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(\gamma, x) d\, vol_N(x).$$

The b_k's depend on the germ of γ (considered as an isometry of O) and on the Riemannian metric.

Testing orbifold-manifold isospectrality

THEOREM *Let O be a Riemannian orbifold with singularities. If O is even-dimensional (respectively, odd-dimensional) and some C-stratum of the singular set is odd-dimensional (respectively, even-dimensional), then O cannot be isospectral to a Riemannian manifold.*

Calculating heat invariants for 2-orbifolds

Let O be an orientable two-dimensional orbifold with k cone points of orders $m_1, \cdots, m_k$. Then the first few terms in the asymptotic expansion are:

- degree -1 term:

$$a_0 = vol(O)$$

- degree 0 term:

$$\frac{\chi(O)}{6} + \sum_{i=1}^{k} \frac{m_i^2 - 1}{12 m_i}$$

- degree 1 term:

$$\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{R_{1212}(m_i^4 + 10 m_i^2 - 11)}{360 m_i}$$

$a_2(O) = \frac{1}{360} \int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$

Teardrops and footballs

Theorem *Within the class of all footballs (good or bad) and all teardrops, the spectral invariant c is a complete topological invariant. I.e., c determines whether the orbifold is a football or teardrop and determines the orders of the cone points.*

Idea of proof

Define a spectral invariant c as 12 times the degree zero term:

$$c = 2\chi(O) + \sum_{i=1}^{k}(m_i - \frac{1}{m_i})$$

For a teardrop with one cone point of order m, we have

$$c(m) = 2 + m + \frac{1}{m}.$$

For a football with cone points of order r and s, we have

$$c(r, s) = r + s + \frac{1}{r} + \frac{1}{s}.$$

When is the invariant an integer?

Suppose $c(m) = c(r, s)$. Then

$$m + 2 = r + s \tag{3}$$

$$\frac{1}{m} = \frac{1}{r} + \frac{1}{s} \tag{4}$$

Contradiction!

Suppose $c(m) = c(r, s)$. Then

$$\begin{aligned} m + 2 &= r + s && (5) \\ \frac{1}{m} &= \frac{1}{r} + \frac{1}{s} && (6) \end{aligned}$$

Contradiction!

Claim: $c(r, s)$ determines r and s

- Read off $r + s$ and $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$
- $c(r, s)$ determines $r + s$ and rs
- $(r - s)^2 = (r + s)^2 - 4rs$, so $c(r, s)$ determines $|r - s|$

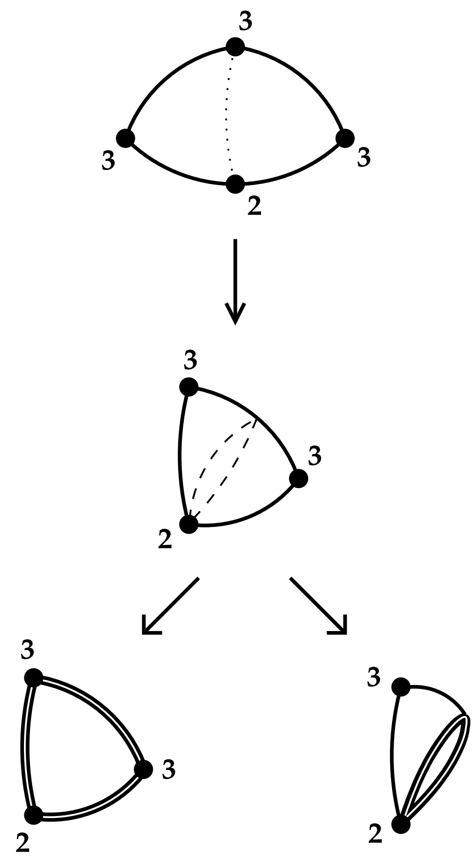
Nonnegative Euler characteristic

THEOREM *Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(O) \geq 0$. The spectral invariant c is a complete topological invariant within C and moreover, it distinguishes the elements of C from smooth oriented closed surfaces.*

Remark We have not included triangular pillows with $\chi(O) < 0$ in the class C. We must add metric assumptions to distinguish among these pillows; however, such assumptions are not necessary to distinguish, say, triangular pillows with $\chi(O) < 0$ from triangular pillows with $\chi(O) > 0$.

Nonorientable 2-orbifolds

Proposition *Within the class of spherical 2-orbifolds of constant curvature $R > 0$ the spectrum determines the orbifold.*



Future Directions

- Are the heat invariants useful in higher dimensions?
- For what classes of orbifolds are the isotropy types spectrally determined?
- What is the relationship between the spectrum of an good orbifold and that of the manifold which finitely covers it?