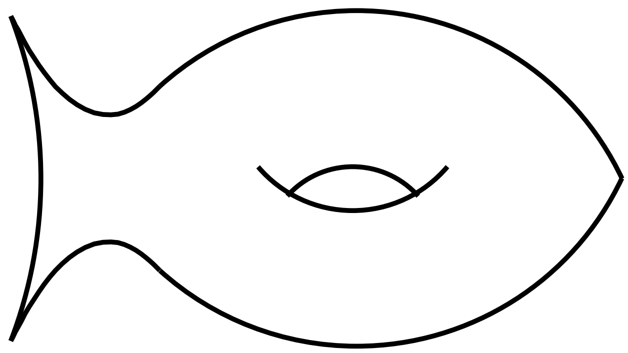
Geometric and Spectral Properties of Compact Riemann Orbisurfaces



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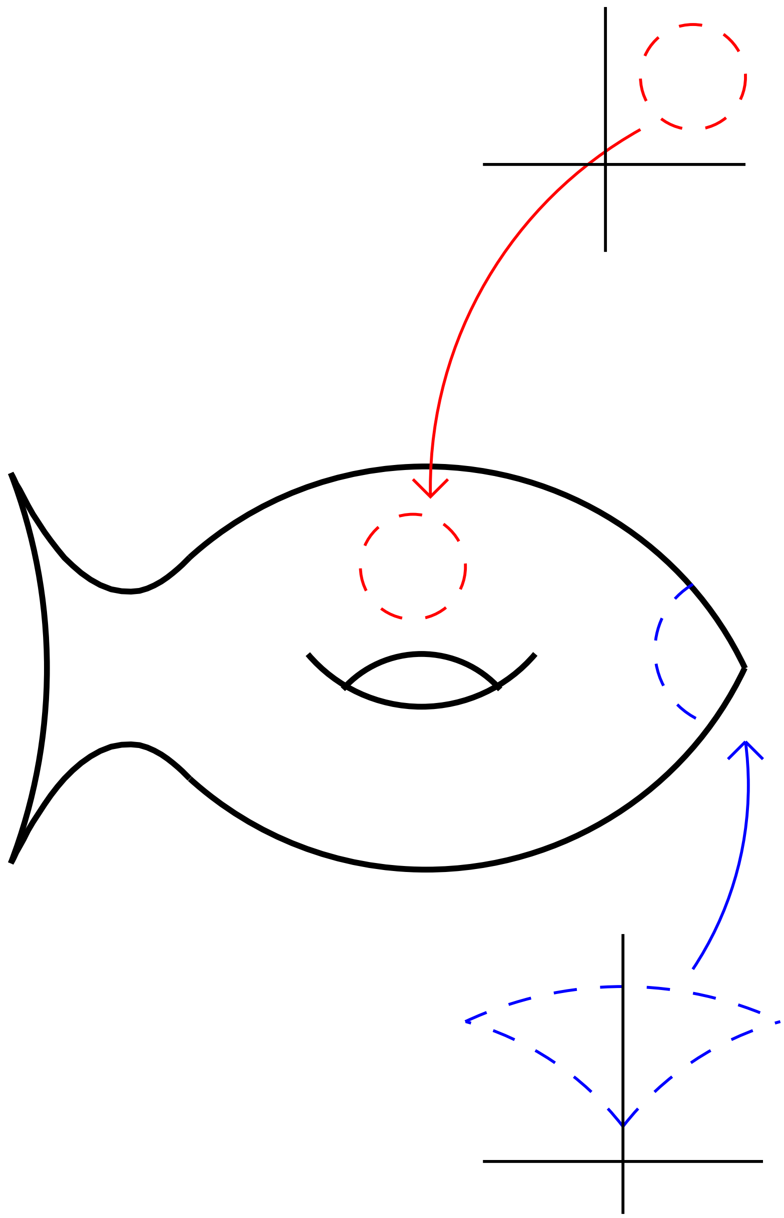
Slides available from
`http://www.math.dartmouth.edu/~edryden`

Hyperbolic Cone-Surfaces

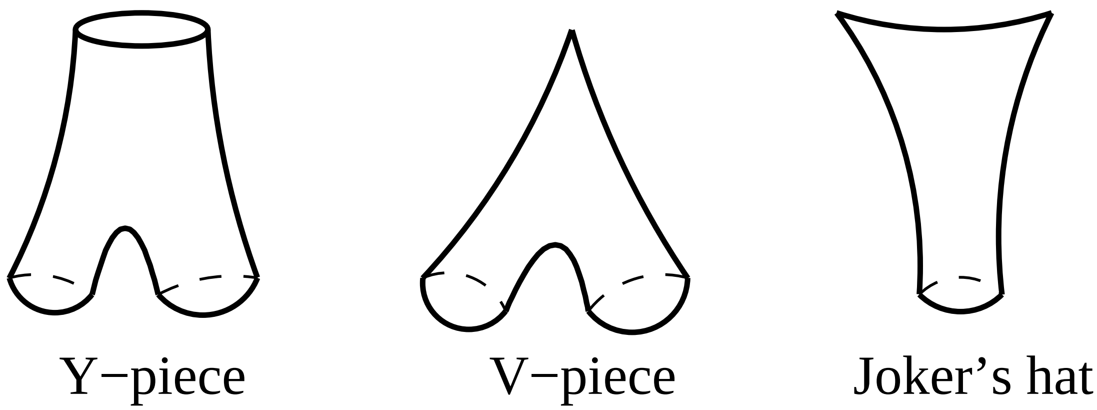


Convention: All cone angles are less than π.

Why study these objects?

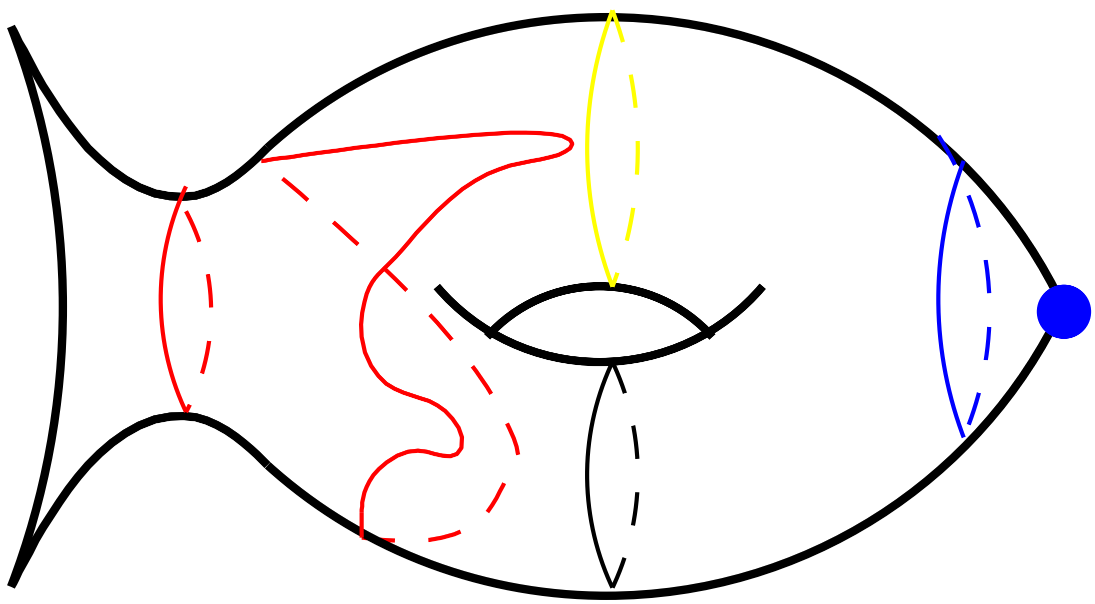
- connections with string theory
- arise in study of three-manifolds
- easiest singular spaces
- orbifolds: visual way to understand a group acting on a space

Pairs of Pants



- Built from hyperbolic geodesic polygons
- Can specify lengths of boundary geodesics and size of cone angles

Geodesic Behavior



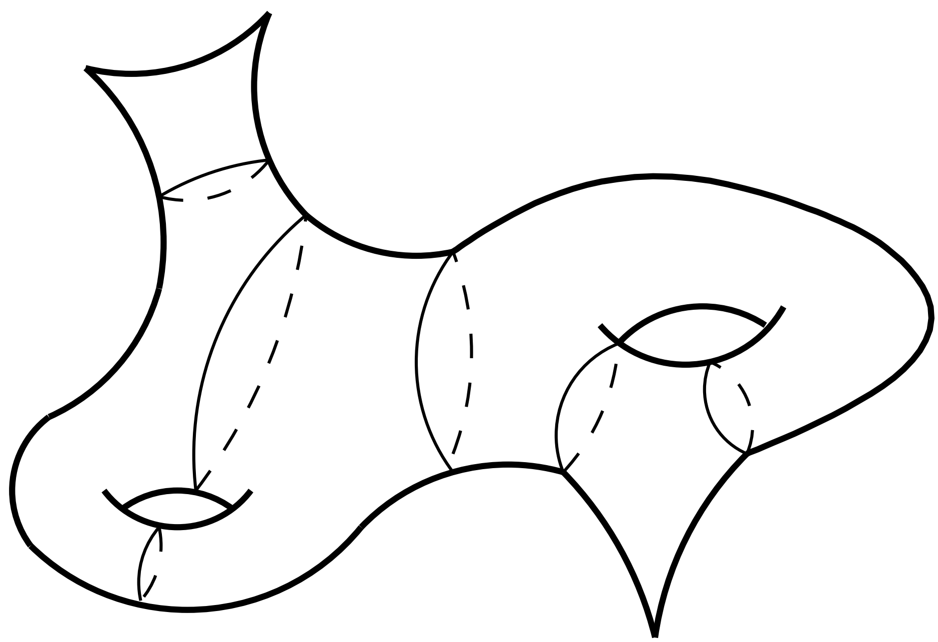
Conventions

Admissible cone-surface: compact, orientable, hyperbolic cone-surface in which all cone angles are less than π

Allowed signatures (g, m, n):

- $(0, 0, k), k \geq 4$
- $(0, 1, l), l \geq 2$
- $(g, m, n) \geq (0, 2, 1), (g, m, n) \neq (1, 0, 0)$

Decomposition into Pairs of Pants



Theorem *(Dianu, extended by D) Let S be an admissible cone-surface of signature $(g, m, n) > (0, 2, 1)$. Then there exists a decomposition of S into $2g - 2 + m$ Y-pieces and n V-pieces.*

What can we say about these partitioning geodesics?

Compact Riemann surfaces:

- tubular neighborhood, called collar, around simple closed geodesic
- width of collar depends uniquely on length of geodesic
- non-intersecting simple closed geodesics have disjoint collars
- widths of collars optimal

Interest in proving similar theorem for hyperbolic orbifolds (Dianu, Gehring and Martin, Matelski): object often to estimate minimal distance between singular points based on order of points

THEOREM *(D-Parlier) Let S be an admissible cone-surface of signature (g, n) with cone points $p_1, \ldots, p_n$ and cone angles $2\varphi_1, \ldots, 2\varphi_n$. Let 2φ be the largest cone angle. Let $\gamma_1, \ldots, \gamma_m$ be disjoint simple closed geodesics on S. Then the following hold.*

1. $m \leq 3g - 3 + n$.
2. *There exist simple closed geodesics $\gamma_{m+1}, \ldots, \gamma_{3g-3+n}$ which together with $\gamma_1, \ldots, \gamma_m$ form a partition of S.*
3. *The collars*

$$\mathcal{C}(\gamma_k) = \{x \in S \mid d(x, \gamma_k) \leq w_k\},$$

where $w_k = \operatorname{arcsinh}(\cos\varphi / \sinh\frac{\gamma_k}{2})$, *and*

$$\mathcal{C}(p_l) = \{x \in S \mid d(x, p_l) \leq v_l\},$$

where $v_l = \operatorname{arccosh}(1/\sin\varphi_l)$, *are pairwise disjoint for* $k = 1, \ldots, 3g - 3 + n$ *and* $l = 1, \ldots, n$.

4. Each $\mathcal{C}(\gamma_k)$ is isometric to the cylinder $[-w_k, w_k] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2(\gamma_k)\cosh^2 \rho dt^2$.
Each $\mathcal{C}(p_l)$ is isometric to a hyperbolic cone $[0, w_l] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \frac{\varphi_l^2}{\pi^2}\sinh^2 \rho dt^2$.

Remarks

- Proof uses topological arguments, information about the behavior of closed curves under homotopy, hyperbolic trigonometry
- The values for these collars are optimal.

Bers' Theorem

There exists a length-bounded partition of every compact Riemann surface of genus $g \geq 2$, where the length bound is a constant depending only on g.

Useful in studying spectral questions: finding rough fundamental domain for action of Teichmüller modular group, finding explicit bound on size of isospectral families, estimates involving Fenchel-Nielsen parameters

Allows significant restriction on lengths of partitioning geodesics

THEOREM (D-Parlier) *Let S be a compact admissible cone-surface of signature (g, n). Then there exists a partition $\mathcal{P}$ of S such that every geodesic in $\mathcal{P}$ has length less than a constant $L_{g,n}$.*

Proof uses area estimates based on polar coordinates

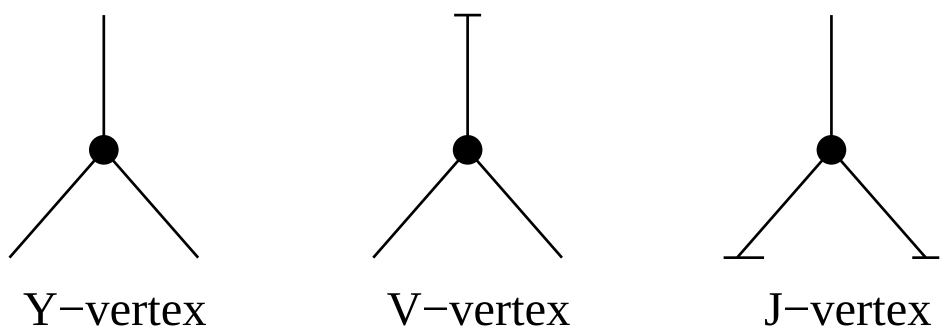
Remarks Get explicit bound for length of each partitioning geodesic from proof:

$$\ell(\gamma_k) < 4\pi k(2g - 2 + n),$$

where γ_k is the kth geodesic in a partition of S. So showed

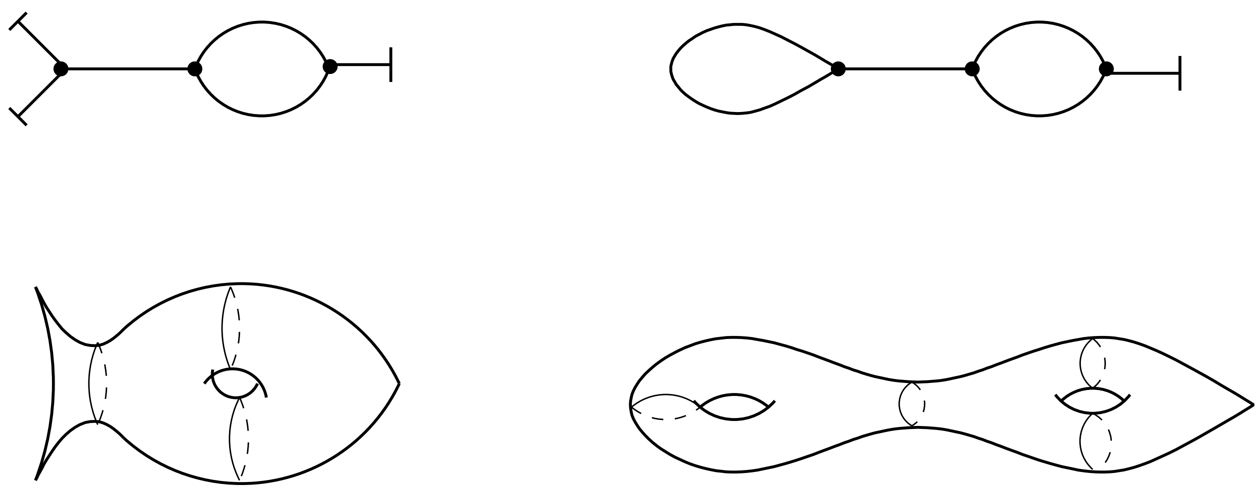
$$L_{g,n} < 4\pi(3g - 3 + n)(2g - 2 + n)$$

Cubic Pseudographs



- 3-regular
- glue unbarred half-edges to form edges
- do not satisfy definition of graph

Pseudographs and Associated Cone-Surfaces



Can we construct all admissible cone-surfaces of signature (g, n) using (marked) cubic pseudographs as underlying combinatorial skeletons?

Construction of Admissible Cone-Surfaces

Fix a (marked) cubic pseudograph G.

Choose

$$\begin{aligned} L &= (\ell_1, \ldots, \ell_{3g-3+n}) \in \mathbb{R}_+^{3g-3+n}, \\ A &= (\alpha_1, \ldots, \alpha_{3g-3+n}) \in \mathbb{R}^{3g-3+n}, \\ N &= (2\varphi_1, \ldots, 2\varphi_n) \in I^n, \end{aligned}$$

where $I = (0, \pi)$.

Associate appropriate pair of pants to each vertex in G.

L and N give length of geodesic/size of cone angle for each boundary component

Do the Twist!

Paste pairs of pants along boundary geodesics of same length

Introduce twists: paste P_i and P_h together along $\gamma_{i\mu}$ and $\gamma_{h\nu}$ via the identification

$$\gamma_{i\mu}(t) = \gamma_{h\nu}(\alpha_k - t) := \gamma_k(t), \qquad t \in \mathbb{S}^1$$

Obtain admissible cone-surface $F(G, L, A, N)$ of signature (g, n) by pasting together all pairs of pants according to G in this manner

(L, A) are Fenchel-Nielsen parameters of cone-surface $F(G, L, A, N)$

Theorem *(D) Let G be a fixed marked cubic pseudograph with $2g-2+n$ vertices, and let N be a fixed set of n cone angles. Then $F(G, L, A, N)$ runs through all admissible cone-surfaces of genus g with n cone points having cone angles in N.*

Shape and Sound

Interested in relationships between geometric and spectral properties of Riemann orbisurfaces

Spectral properties: what can we learn from the eigenvalue spectrum of the Laplace operator as it acts on smooth functions on a given Riemann orbisurface?

Milnor: pair of isospectral non-isometric 16-dimensional flat tori

Recent interest in non-smooth case (orbifolds)

Spectral Theory on Orbifolds

Define smooth function and Laplace operator on orbifolds locally

THEOREM *(Chiang) Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*
2. *Each eigenvalue λ_i has finite multiplicity.*
3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta\phi_i = \lambda_i\phi_i$.*

Huber's Theorem

THEOREM *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

length spectrum is sequence of all lengths of all oriented closed geodesics on surface, arranged in ascending order

THEOREM *(D) If two admissible Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Finiteness of Isospectral Sets

McKean: only finitely many compact Riemann surfaces have a given Laplace spectrum

THEOREM *(D) Let O be an admissible Riemann orbisurface of genus $g \geq 1$. In the class of compact orientable hyperbolic orbifolds with cone points of order three and higher, there are only finitely many members which are isospectral to O.*

Remarks

- no dimension restriction on orbifolds isospectral to O
- no Riemann surfaces can be isospectral to O

Future Directions

- Explicit bound on size of isospectral sets
- Examples
- What properties of orbifolds are spectrally determined?
- Better understanding of geometry of cone-surfaces, e.g. injectivity radius