Upper bounds for invariant eigenvalues of the Laplacian

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The Plan

1. Historical Motivation
2. Two Dimensions
3. Higher Dimensions:
   - S^n with an action of $O(n)$
   - Manifolds with an action of a Lie group

Setup

M: compact connected Riemannian manifold of dimension $n \geq 2$

Laplacian: Δ = -div grad

Spec(g) =
$\{0 = \lambda_0(g) < \lambda_1(g) \leq \ldots \leq \lambda_k(g) \leq \ldots\}$

Question Consider the kth eigenvalue, normalised as a functional

$$g \to \lambda_k(g)\mathrm{Vol}(g)^{2/n}$$

on the space of Riemannian metrics. What are critical/extremal metrics for this functional?

Answers for Surfaces

S^2 [Hersch] :

$$\lambda_1(g)\mathrm{Vol}(g) \leq 8\pi = \lambda_1(g_{\mathrm{can}})\mathrm{Vol}(g_{\mathrm{can}})$$

Orientable surface of genus γ [Yang-Yau]:

$$\lambda_1(g)\mathrm{Vol}(g) \leq 8\pi \left[\frac{\gamma+3}{2}\right]$$

Surface of genus γ [Korevaar]: There exists a universal constant $C > 0$ such that for all $k > 0$,

$$\lambda_k(g) \leq Ck(\gamma+1).$$

Answers in Higher Dimensions

THEOREM *[Colbois-Dodziuk]:*

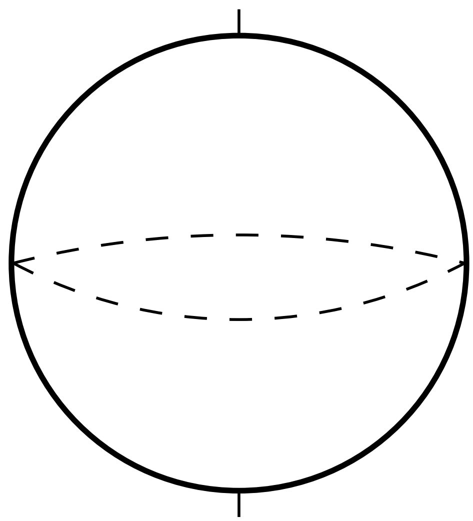
$$\sup_g \{\lambda_1(g)\, Vol(g)^{2/n}\} = \infty$$

To get finite bounds, we need to add some restrictions...

- conformal class (Korevaar, El Soufi-Ilias)
- projective Kähler metrics (Bourguignon-Li-Yau)
- symplectic or Kähler metrics (L. Polterovich)
- invariance under isometries (Abreu-Freitas)

S^1 acts on S^2

Consider S^2 with metrics which are smooth, have total area 4π, and are S^1-invariant.



Denote the invariant eigenvalues by $\lambda_k^{\text{inv}}(g)$.

THEOREM *[Abreu-Freitas]: In this setting, $\lambda_1^{\text{inv}}(g)$ can be any number strictly between 0 and ∞.*

Do more restrictions help get bounds?

Fixed Gauss curvature at poles: still have $0 < \lambda_1^{\mathrm{inv}}(g) < \infty$

Metrics embedded in $\mathbb{R}^3$:

$$\lambda_k^{\mathrm{inv}}(g) < \frac{1}{2}\xi_k^2$$

and in particular

$$\lambda_1^{\mathrm{inv}}(g) < \frac{1}{2}\xi_1^2 \approx 2.89$$

Can characterize the supremum geometrically

Questions, Questions, Questions

Questions

- How do the invariant eigenvalues of a manifold behave under general group actions?
- Is being embedded essential to bounding $\lambda_k^{\mathrm{inv}}(g)\mathrm{Vol}(g)$?
- If we find critical metrics, to what do they correspond geometrically?

Dimension 2

Nontrivial S^1-actions exist on

- sphere
- torus
- projective plane
- Klein bottle

PROPOSITION *[Colbois-D-El Soufi]: Within the class of smooth S^1-invariant metrics g on T^2 which correspond to an embedding of T^2 in $\mathbb{R}^3$,*

$$\sup_g \{\lambda_1^{inv}(g) Vol(g)\} = \infty.$$

Remark The argument also works for a general torus $T^{n+1} = S^1 \times S^n$.

Higher Dimensions

Consider $O(n)$-invariant metrics on S^n embedded in $\mathbb{R}^{n+1}$, of volume 1.

THEOREM *[Colbois-D-El Soufi]: For all k,*

$$\lambda_k^{inv}(g) < \lambda_k^{inv}(D^n).$$

Furthermore, there exists a sequence of embeddings of S^n in $\mathbb{R}^{n+1}$ with

$$\lambda_k^{inv}(g_i) \to \lambda_k^{inv}(D^n),$$

but $\lambda_k^{inv}(D^n)$ is not attained by any smooth metric on S^n.

Lie Groups

Assumptions

- $dim(M) \geq 3$
- G: *Lie group of dimension* ≥ 1 *acting on* M *by isometries*
- $dim(M/G) = d \geq 1$

Theorem *[Colbois-D-El Soufi]: Let* (M, g_0) *and* G *be as above. Then*

$$\sup_g \{\lambda_1^{inv}(g)\, Vol(g)^{2/n}\} = \infty,$$

where the metrics g *are* G*-invariant and conformal to* g_0.

Discrete Lie Groups

THEOREM *[Korevaar]: Let (M, g_0) be a compact Riemannian manifold and $G < Isom(M)$ a discrete group such that the quotient M/G is orientable. Then*

$$\sup_g \{\lambda_k^{inv}(g) Vol(g)^{2/n}\} \leq C_n(g_0) k^{2/n},$$

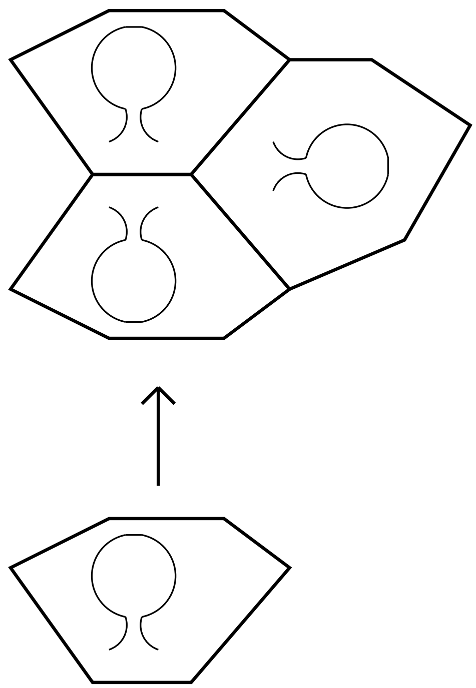
where g is a G-invariant metric conformal to g_0, and C depends only on n and g_0.

Removing the conformal requirement

THEOREM *[Colbois-D-El Soufi]: Let* (M, g_0) *be a compact Riemannian manifold of dimension* $n \geq 3$*, and* $G < Isom(M)$ *a discrete group. Then*

$$\sup_g \{\lambda_1^{inv}(g) Vol(g)^{2/n}\} = \infty.$$

Idea of Proof



Future Directions

- Can we construct G-invariant metrics, G discrete, such that $\lambda_1(g)\mathrm{Vol}(g)^{2/n}$ gets arbitrarily large?
- What happens if we look at invariant p-forms, $p > 0$?
- Can we say more about critical invariant metrics?