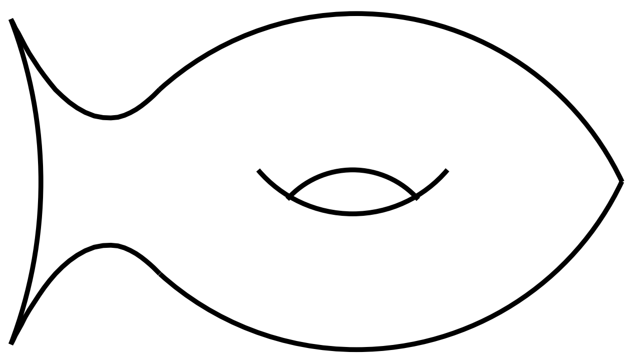
Listening to Orbifolds: What does the Laplace spectrum tell us?



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Slides available from
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The Plan:

1. Historical Motivation
2. Vibrating Strings
3. Drums and Manifolds
4. Orbifolds

Historical Motivation

Chemistry: identify elements by spectral "fingerprints"

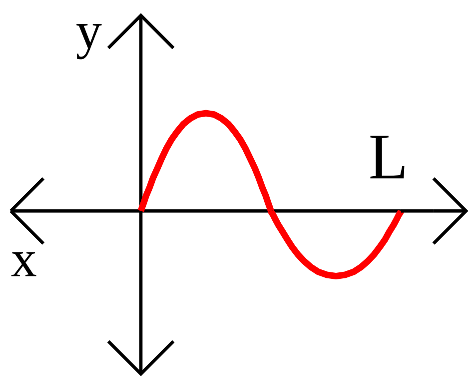
Physics: development of quantum mechanics

Mathematics: how are knowledge of structure and knowledge of spectrum related?

Vibrating Strings

Setup: string of length L with uniform density and tension, fixed endpoints

Pluck the string:



Describe motion of string with function $f(x, t)$

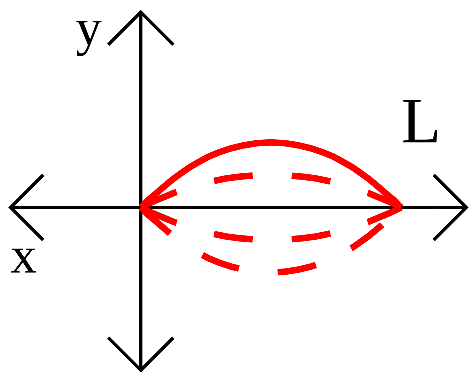
Constraints on $f(x, t)$:

- $f(0, t) = 0$
- $f(L, t) = 0$
- $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$

Solving the Wave Equation

Look for waveforms, i.e. solutions $f(x,t)$ such that

$$f(x,t) = g(x)h(t)$$



$g(x)$ gives shape
$h(t)$ measures amplitude

Substitute solution into wave equation:

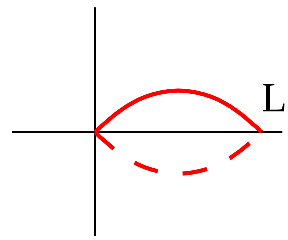
$$\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)} = \lambda$$

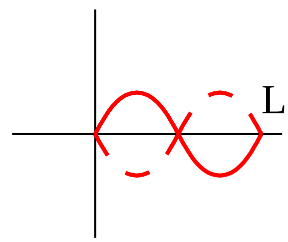
so

$$T(h) = \lambda h, \qquad T(g) = \lambda g$$

Waveforms

Specific waveforms oscillate at specific frequencies





Sound of string of length L is overtone sequence $\frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \ldots$

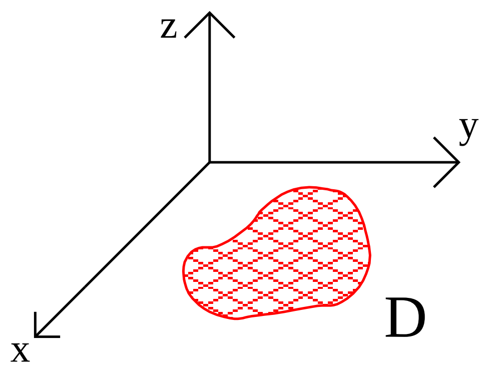
Waveforms form basis for vector space of motion functions $f(x,t)$

Can calculate sound from length

Can "hear" the length of a string!

Can one hear the shape of a drum?

D = compact domain in Euclidean plane



Describe motion with function $f(x, y, t)$

Constraints on $f(x, y, t)$:

- $f(x_0, y_0, t) = 0$ for all (x_0, y_0) on boundary
- $-\frac{\partial^2 f}{\partial t^2} = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} := \Delta f$

Waveforms Again

Sound of drum given by list of frequencies associated to waveforms

$$f(x, y, t) = g(x, y)h(t)$$

Substitute solution into wave equation:

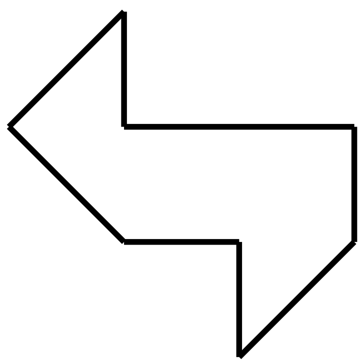
$$\frac{\Delta g}{g} = -\frac{h''}{h} = \lambda$$

Frequencies of vibration = Eigenvalues of Δ on D

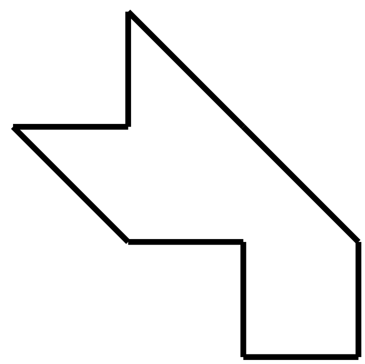
Cannot explicitly calculate list of frequencies in general

Can hear area and perimeter of drumhead

You cannot hear the shape of a drum!



D_1



D_2

Manifolds

M = compact Riemannian manifold
$\Delta = -div\ grad$

Big Question: How much geometric information about M is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- M = surface:
  Euler characteristic, hence genus

- round spheres characterized by spectra
- isospectral nonisometric Riemann surfaces
- isospectral nonisometric planar domains

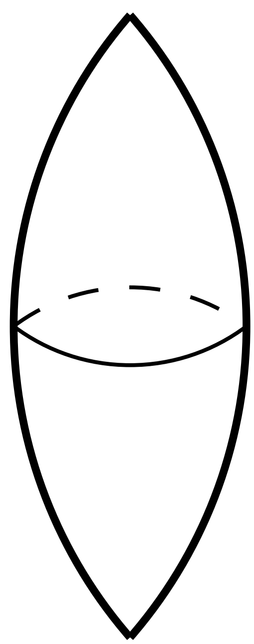
$$
\begin{tikzcd}
 & & M \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & M_1 \arrow[dl] \arrow[dr] & & M_2 \arrow[dl] \arrow[dr] & \\
O_1 \arrow[drr] & & M_0 \arrow[d] & & O_2 \arrow[dll] \\
 & & O_0 & &
\end{tikzcd}
$$

What is an orbifold?

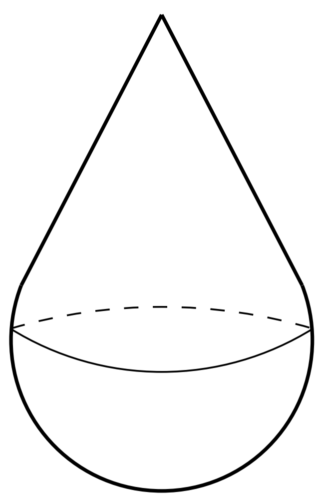
EXAMPLES

1. Manifolds
2. M/Γ, where Γ is a group acting "nicely" on a manifold M

Let $M = S^2$, and let Γ be the group of rotations of order 3 about the north-south axis. Then M/Γ is a $(3, 3)$-football.



3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Riemannian Orbifolds

Construction of Riemannian metric on O:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\mathrm{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$
- Each eigenvalue λ_i has finite multiplicity.
- Orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions

Inverse Spectral Geometry of Orbifolds

- Gordon, Webb and Wolpert (1992): used orbifolds in construction of drum examples

- Gordon and Rossetti (2003): middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

- Gordon, Greenwald, Webb and Zhu (2003): spectral invariant for footballs and teardrops

- Shams, Stanhope and Webb: there exist arbitrarily large (but always finite) isospectral sets, where each element in a given set has points of distinct isotropy

Listening to Orbifolds

O = compact Riemannian orbifold
$\Delta = -div\ grad$ (locally)

Big Question: How much geometric information about O is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- orbisurfaces: genus???
- isospectral nonisometric Riemann orbisurfaces???

Tools in Dimension 2

O: orbisurface with s cone points of orders $m_1, \ldots, m_s$

Define the (orbifold) Euler characteristic of O to be

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s} (1 - \frac{1}{m_j}).$$

THEOREM *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O).$$

Euler characteristic is spectrally determined, but unknown if spectrum determines genus

Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of Riemann orbisurfaces. Specifically, we show

Theorem *(D.) Let O be a compact Riemann orbisurface with genus $g \geq 1$. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Huber's Theorem for Compact Riemann Surfaces

THEOREM *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

length spectrum: sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order

An Analog of Huber's Theorem

THEOREM *(D.-Strohmaier) If two compact Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra and a sum involving the orders of the cone points. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Selberg Trace Formula for Compact Riemann Orbisurfaces

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr$$

$$+ \sum_{\substack{\{R\} \\ \text{elliptic}}} \frac{1}{2m(R) \sin\theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1+e^{-2\pi r}} h(r) dr$$

$$+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)]$$

Sketch of Proof:

Use appropriate version of Selberg Trace Formula

- Plug in a “good” test function for $h(r)$
- Know volume from Weyl’s asymptotic formula
- Read off lengths of geodesics from singular support of wave trace
- Left with elliptic summand involving orders of cone points

Explicit Bounds

THEOREM *(Buser) Let S be a compact Riemann surface of genus $g \geq 2$. At most e^{720g^2} pairwise non-isometric compact Riemann surfaces are isospectral to S.*

No g-independent upper bound is possible

Brooks, Gornet, and Gustafson examples: cardinality of set grows faster than polynomially in g

Bounds for Riemann Orbisurfaces

- Cubic pseudographs (D.)
- Fenchel-Nielsen parameters (D.)
- Collar theorem (D.-Parlier)
- Bers' theorem (D.-Parlier)
- Understanding of geodesic behavior

Future Directions

- How do geodesics on orbifolds behave?
- For what classes of orbifolds are the isotropy types spectrally determined?
- What is the relationship between the spectrum of a Riemann orbisurface and that of the Riemann surface which finitely covers it?