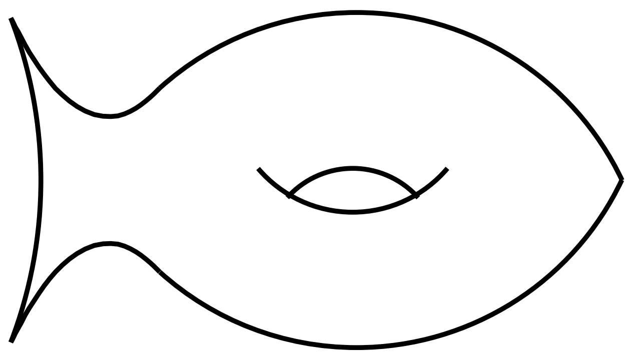
Isospectrality of Compact Riemann Orbisurfaces



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Recent developments in spectral geometry
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Slides available from
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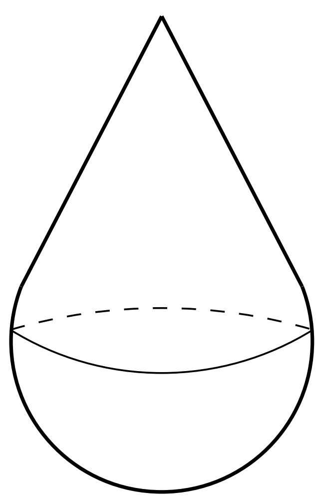
What is an orbifold?

Examples

1. Manifolds
2. M/Γ, where Γ is a group acting properly discontinuously on a manifold M
3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Why are orbifolds of interest?

1. Visual way to understand group acting on a space
2. Study of 3-manifolds
3. Easiest singular spaces
4. Crystallography
5. String theory

Riemannian Orbifolds

Construction of Riemannian metric on O analogous to manifold case

Structures must be invariant under local group actions.

Riemann orbisurface: orientable, two-dimensional Riemannian orbifold with metric of constant curvature -1

Use local definitions (e.g. function, Laplacian)

Results of local analysis hold, but global results may not hold or take new form.

THEOREM *(Chiang) Let O be a compact Riemannian orbifold.*

1. *The set of eigenvalues λ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on O, denoted $Spec(O)$.*

2. *Each eigenvalue λ_i has finite multiplicity.*

3. *There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta\phi_i = \lambda_i \phi_i$.*

Spectral Theory of Orbifolds

- Gordon and Rossetti (2003): middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

- Farsi (2001): Weyl's asympotic formula holds for orbifolds

- Gordon, Greenwald, Webb and Zhu: spectral invariant which, within the class of all footballs and teardrops, determines the orbisurface

- Shams, Stanhope and Webb: there exist arbitrarily large (but always finite) isospectral sets, where each element in a given set has points of distinct isotropy

Weyl's Asymptotic Formula

THEOREM *(Farsi) Let O be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum* $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. *Then for the function* $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ *we have*

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O)\frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$.

The Laplace spectrum determines an orbifold's dimension and volume.

Tools in Dimension 2

O: orbisurface with s cone points of orders $m_1, \ldots, m_s$

Define the (orbifold) Euler characteristic of O to be

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s}(1 - \frac{1}{m_j}).$$

Theorem *(Gauss-Bonnet) Let O be a two-dimensional Riemannian orbifold. Then*

$$\int_O K dA = 2\pi\chi(O).$$

Obstructions to Isospectrality

PROPOSITION *Let O be a compact Riemann orbisurface of genus $g_0 \geq 0$ with k cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be a compact orientable hyperbolic orbifold of genus $g_1 \geq g_0$ with l cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k+h)$, then O is not isospectral to O'.*

COROLLARY *Fix $g \geq 0$. Let O be a compact Riemann orbisurface of genus g with k cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let O' be a compact orientable hyperbolic orbifold of genus g with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then O is not isospectral to O'.*

Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of Riemann orbisurfaces. Specifically, we show

THEOREM *Let O be a compact Riemann orbisurface with genus $g \geq 1$. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Huber's Theorem for Compact Riemann Surfaces

THEOREM *(Huber) Two compact Riemann surfaces of genus $g \geq 2$ have the same spectrum of the Laplacian if and only if they have the same length spectrum.*

length spectrum: sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order

A Partial Analog of Huber's Theorem

THEOREM *If two compact Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra and the orders of their cone points, up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.*

Selberg Trace Formula for Compact Riemann Orbisurfaces

$$\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr$$

$$+ \sum_{\substack{\{R\} \\ \text{elliptic}}} \frac{1}{2m(R)\sin\theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1+e^{-2\pi r}} h(r) dr$$

$$+ \sum_{\substack{\{P\} \\ \text{hyperbolic}}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)]$$

Sketch of Proof:

Use appropriate version of Selberg Trace Formula

- Know volume from Weyl's asymptotic formula
- Determine elliptic summand up to finitely many possibilities
- Read off lengths

Finiteness of Isospectral Sets

THEOREM *Let O be a compact Riemann orbisurface with genus $g \geq 1$. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to O.*

Sketch of Finiteness Proof:

S: class of compact orientable hyperbolic orbifolds which are isospectral to O

- any member of S determined by its fundamental group Γ
- to specify Γ, suffices to specify single, double, triple traces of generating set
- using extended Huber's theorem, can bound traces of hyperbolic conjugacy classes by $2\cosh D, 2\cosh 2D, 2\cosh 3D$
- common upper bound on diameter of any orbisurface in S
- finitely many choices for trace of elliptic element in Γ

Explicit Bounds

THEOREM *(Buser) Let S be a compact Riemann surface of genus $g \geq 2$. At most e^{720g^2} pairwise non-isometric compact Riemann surfaces are isospectral to S.*

No g-independent upper bound is possible

Brooks, Gornet, and Gustafson examples: cardinality of set grows faster than polynomially in g

Future Directions

- For what classes of orbifolds are the isotropy types spectrally determined?
- What is the relationship between the spectrum of a Riemann orbisurface and that of the Riemann surface which finitely covers it?
- How do geodesics on orbifolds behave?