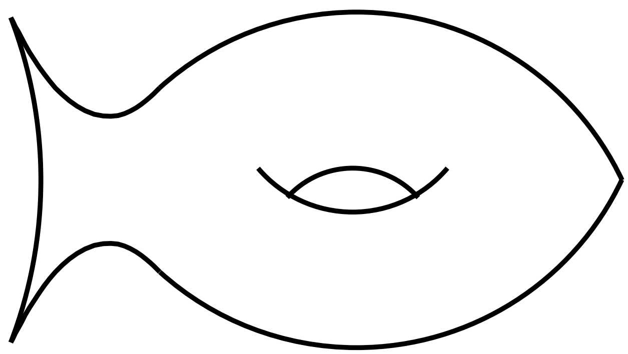
Collars and Partitions of Hyperbolic Cone-Surfaces



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Slides available from
`http://www.math.mcgill.ca/~dryden`

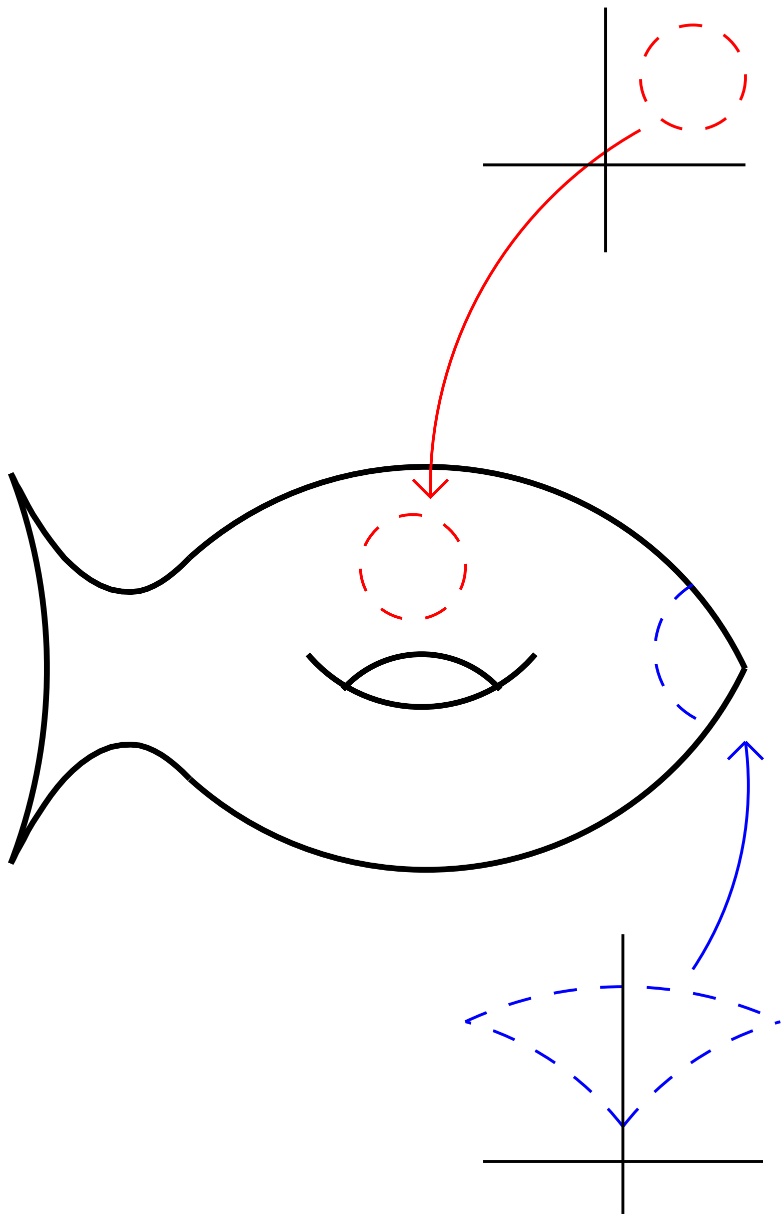
Let S be a compact hyperbolic Riemann surface of genus $g \geq 2$.

Collar Theorem *Every simple closed geodesic on S has a tubular neighborhood, and the width of this neighborhood depends uniquely on the length of the geodesic. Non-intersecting simple closed geodesics have disjoint collars.*

Bers' Theorem *A partition of S into pairs of pants can be chosen such that the lengths of the partitioning geodesics are bounded by a constant depending only on the genus.*

Hyperbolic Cone-Surfaces

Definition A hyperbolic cone-surface is a 2-dimensional manifold which can be triangulated by hyperbolic geodesic triangles.



Assumptions

1. compact

2. orientable

3. all cone angles are less than π

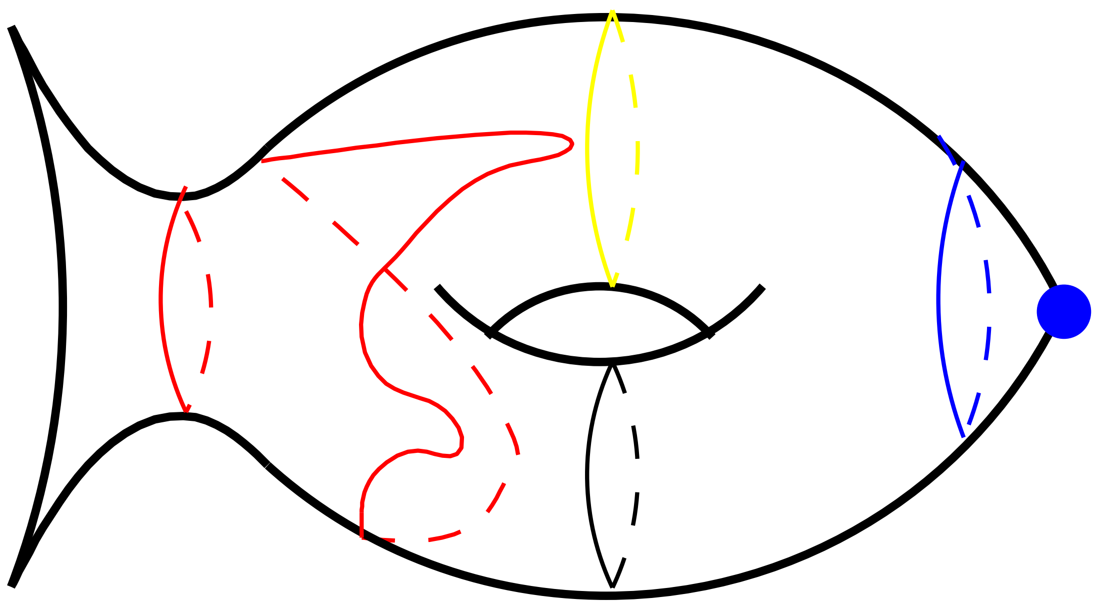
4. $(g, n) \geq (0, 4)$

We call a hyperbolic cone-surface satisfying these assumptions an *admissible* cone-surface.

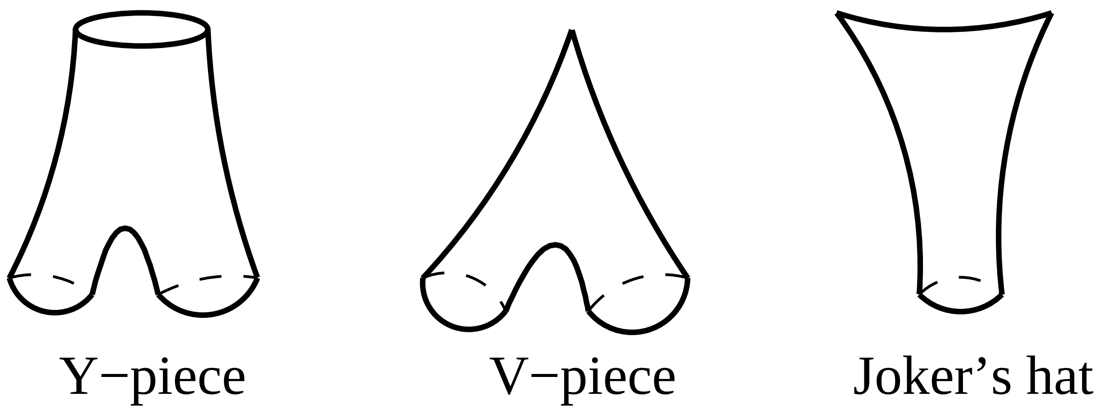
Why study these objects?

- connections with string theory
- arise in study of three-manifolds
- easiest singular spaces
- orbifolds: visual way to understand a group acting on a space

Geodesic Behavior



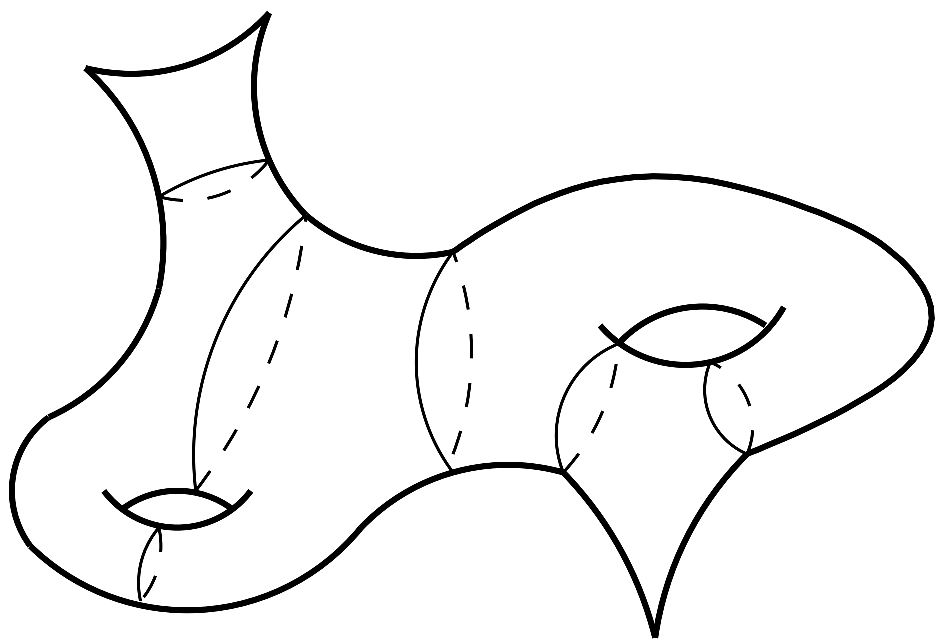
Pairs of Pants



- Built from hyperbolic geodesic polygons
- Can specify lengths of boundary geodesics and size of cone angles

Decomposition into Pairs of Pants

Definition Let S be an admissible cone-surface. A *partition* P of S is a set of simple closed geodesics $\{\gamma_1, \ldots, \gamma_m\}$ such that $S \setminus P$ is a set of pairs of pants.



THEOREM *(D-Parlier) Let S be an admissible cone-surface of signature (g, n) with cone points $p_1, \ldots, p_n$ and cone angles $2\varphi_1, \ldots, 2\varphi_n$. Let 2φ be the largest cone angle. Let $\gamma_1, \ldots, \gamma_m$ be disjoint simple closed geodesics on S. Then the following hold.*

1. *$m \leq 3g - 3 + n$.*
2. *There exist simple closed geodesics $\gamma_{m+1}, \ldots, \gamma_{3g-3+n}$ which together with $\gamma_1, \ldots, \gamma_m$ form a partition of S.*
3. *The collars*

$$\mathcal{C}(\gamma_k) = \{x \in S \mid d(x, \gamma_k) \leq w_k\},$$

where $w_k = \operatorname{arcsinh}(\cos\varphi / \sinh\frac{\gamma_k}{2})$, and

$$\mathcal{C}(p_l) = \{x \in S \mid d(x, p_l) \leq v_l\},$$

where $v_l = \operatorname{arccosh}(1/\sin\varphi_l)$, are pairwise disjoint for $k = 1, \ldots, 3g - 3 + n$ and $l = 1, \ldots, n$.

4. Each $\mathcal{C}(\gamma_k)$ is isometric to the cylinder $[-w_k, w_k] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \ell^2(\gamma_k)\cosh^2\rho dt^2$.
Each $\mathcal{C}(p_l)$ is isometric to a hyperbolic cone $[0, w_l] \times \mathbb{S}^1$ with the Riemannian metric $ds^2 = d\rho^2 + \frac{\varphi_l^2}{\pi^2}\sinh^2\rho dt^2$.

Remarks

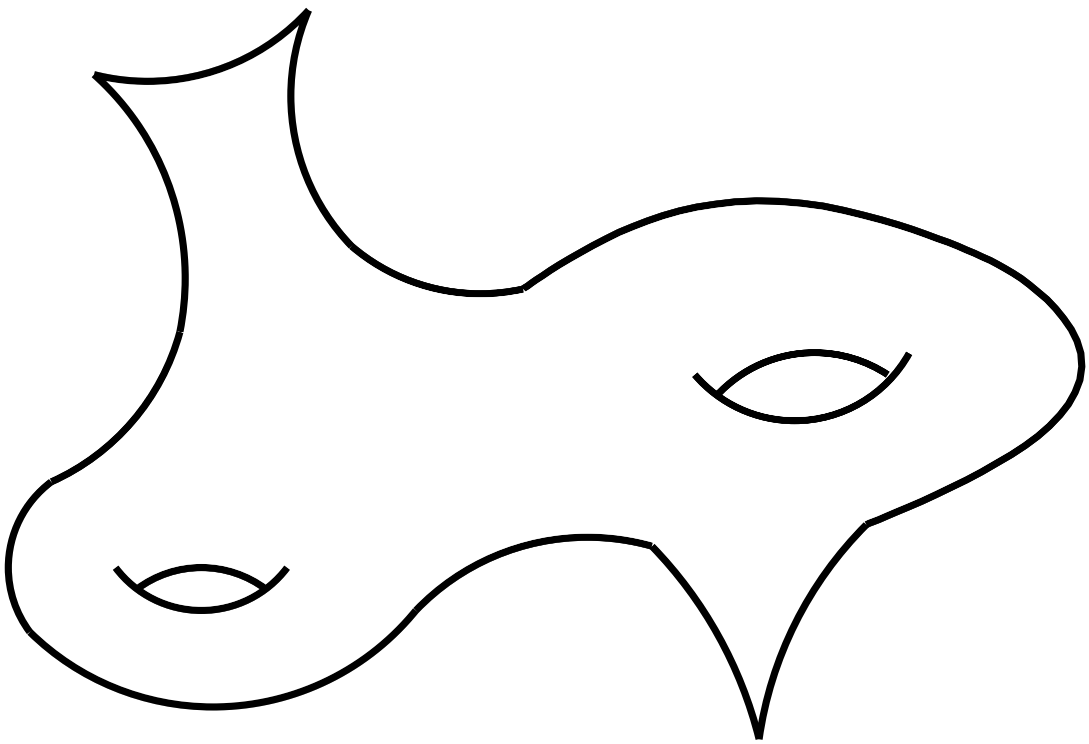
- Proof uses topological arguments, information about the behavior of closed curves under homotopy, hyperbolic trigonometry
- The values for these collars are optimal.

Bers' Theorem for Admissible Cone-Surfaces

THEOREM *(D-Parlier) Let S be an admissible cone-surface of signature (g, n). Then there exists a partition $\mathcal{P}$ of S such that every geodesic in $\mathcal{P}$ has length less than a constant $L_{g,n}$.*

Proof uses area estimates based on polar coordinates

Idea of Proof



Remarks Get explicit bound for length of each partitioning geodesic from proof:

$$\ell(\gamma_k) < 4\pi k(2g - 2 + n),$$

where γ_k is the kth geodesic in a partition of S. So showed

$$L_{g,n} < 4\pi(3g - 3 + n)(2g - 2 + n).$$