Isoperimetric inequalities
Upper bounds on eigenvalues for manifolds
Metrics invariant under a group action
Submanifolds

Upper Bounds on Eigenvalues of the Laplacian: Surfaces and Beyond

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Outline

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A little history
Rayleigh quotients

The Original Isoperimetric Inequality

- The Problem of Queen Dido: maximize the size of Carthage

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The Original Isoperimetric Inequality

- The Problem of Queen Dido: maximize the size of Carthage
- What about *closed* curves?

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The Original Isoperimetric Inequality

- The Problem of Queen Dido: maximize the size of Carthage
- What about *closed* curves?
  - planar
  - simple
  - fix length *L*, maximize area *A*

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The Original Isoperimetric Inequality

- The Problem of Queen Dido: maximize the size of Carthage
- What about *closed* curves?
  - planar
  - simple
  - fix length L, maximize area A
  - "The" isoperimetric inequality:

$$L^2 \geq 4\pi A$$

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Generalizations

- $\mathbb{R}^n$: minimize surface area among domains with fixed volume

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Generalizations

- $\mathbb{R}^n$: minimize surface area among domains with fixed volume
- Mathematical physics: a physical quantity is extremal for a circular or spherical domain

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An example

Setup:

- domain $D \subset \mathbb{R}^2$
- $f : D \to \mathbb{R}$, a smooth function which equals zero on the boundary of D
- $\Delta f := \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$

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Seek solutions to $\Delta f = \lambda f$

Especially interested in λ_1

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The Rayleigh quotient for domains

Theorem

Let D be a domain with Δ acting on piecewise smooth, nonzero functions f which are zero on the boundary of D, and with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$. For any such f,

$$\lambda_1 \leq \frac{\int_D |\nabla f|^2}{\int_D f^2},$$

with equality if and only if f is an eigenfunction of λ_1.

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Minima of the Rayleigh quotient

Theorem (Rayleigh, Faber-Krahn)

Among all domains $D \subset \mathbb{R}^2$ with fixed area, the infimum of the Rayleigh quotient attains a minimum if and only if D is a circular disk.

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Higher-dimensional analog: Rayleigh quotient attains minimum iff $D \subset \mathbb{R}^n$ is sphere

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The Rayleigh quotient for manifolds

Setup:

- (M, g), compact Riemannian manifold
- Δ, Laplace operator on (M, g)
- Eigenvalues of Δ are

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

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The Rayleigh quotient for manifolds

Setup:

- (M, g), compact Riemannian manifold
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- Eigenvalues of Δ are

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

- Rayleigh quotient:

$$\lambda_1(M) = \inf_{f \in \mathcal{F}_1} \frac{\int_M |\nabla f|^2}{\int_M f^2},$$

where $\mathcal{F}_1$ is set of smooth nonzero functions on M orthogonal to the constant functions

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Hersch's Theorem

Theorem (Hersch)

Consider the sphere S^2 equipped with any Riemannian metric g. We have

$$\lambda_1 \, Vol(g) \leq 8\pi,$$

with equality only in the case of the constant curvature metric.

Idea of proof: Move S^2 to its center of mass, and use coordinate functions as test functions in the Rayleigh quotient.

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Dimension 2
Higher dimensions

Compact orientable surfaces

Theorem (Yang-Yau)

Let (M, g) be a compact orientable surface of genus γ. Then

$$\lambda_1(g) \, Vol(g) \le 8\pi \left\lfloor \frac{\gamma + 3}{2} \right\rfloor .$$

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Generalized to nonorientable surfaces by Li-Yau

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What's changed?

Theorem (Korevaar)

Let (M, g) be a compact orientable surface of genus γ, and let $C > 0$ be a universal constant. For every integer $k \geq 1$,

$$\lambda_k(g) Vol(g) \leq C(\gamma + 1)k.$$

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Open questions abound, e.g., *optimal* bound for λ_2 on Klein bottle or surface of genus 2

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Dimension 3

Bleecker: For every $n \geq 3$, the sphere S^n admits metrics of volume one with λ_1 arbitrarily large.

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Dimension 2
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Dimension 3

Bleecker: For every $n \geq 3$, the sphere S^n admits metrics of volume one with λ_1 arbitrarily large.

Theorem (Colbois-Dodziuk)

Let (M^n, g) be a compact, closed, connected manifold of dimension at least three. Then

$$\sup \lambda_1(g) \, Vol(g)^{2/n} = \infty,$$

where the supremum is taken over all Riemannian metrics g on M.

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Idea of proof

- Use Bleecker's result: take (S^n, g_0) such that $\mathrm{Vol}(S^n, g_0) = 1$ and $\lambda_1(g_0) \geq k + 1$, where k is a large constant

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- Use Bleecker's result: take (S^n, g_0) such that $\text{Vol}(S^n, g_0) = 1$ and $\lambda_1(g_0) \geq k + 1$, where k is a large constant
- Form connected sum of S^n and M

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- Connected sum is diffeomorphic to M, contains submanifold Ω naturally identified with $S^n \setminus B_\rho$

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- Take arbitrary metric g_1 on M whose restriction to Ω equals g_0 restricted to Ω, make g_1 really small on most of $M \setminus \Omega$ without changing it on Ω
- M "looks like" (S^n, g_0), and λ_1 for modified g_1 is like $\lambda_1(g_0)$

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Dimension 2
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Where do we go from here?

To study extremal properties of the Laplace spectrum in dimensions greater than two, we must add more constraints!

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Where do we go from here?

To study extremal properties of the Laplace spectrum in dimensions greater than two, we must add more constraints!

- intrinsic constraints: restrict to conformal class of metrics, to projective Kähler metrics, to metrics which preserve the symplectic or Kähler structure, etc.
- extrinsic constraints: mean curvature (Reilly's inequality)

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Invariant metrics on spheres
Other invariant metrics

Back to the 2-sphere

Tweak Hersch's assumptions:

- consider the subset of S^1-invariant metrics
- let Δ act on S^1-invariant functions
- resulting eigenvalues denoted $\lambda_k^{S^1}$

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Bound is attained by the union of two disks of equal area

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What happens for higher-dimensional spheres?

- replace S^1 by $O(n)$

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What happens for higher-dimensional spheres?

- replace S^1 by $O(n)$
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What happens for higher-dimensional spheres?

- replace S^1 by $O(n)$
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- consider $O(n)$-invariant metrics on S^n arising from *embeddings* of S^n in $\mathbb{R}^{n+1}$

Theorem (Colbois-D-El Soufi)

Let (S^n, g) be as above, with $Vol(g) = 1$. Then, for all $k \in \mathbb{Z}$,

$$\lambda_k^{O(n)}(g) < \lambda_k^{O(n)}(D^n) Vol(D^n)^{2/n},$$

where D^n is the Euclidean n-ball of volume $1/2$.

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What about *any* manifold, not just spheres?

- replace S^n by ccc manifold M of dimension $n \geq 3$
- replace $O(n)$ by finite subgroup G of group of diffeomorphisms acting on M
- let Δ act on G-invariant functions
- consider G-invariant metrics on M

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Then $\lambda_1^G(g)\text{Vol}(g)^{2/n}$ is unbounded!

Proof: apply Colbois-Dodziuk "equivariantly"

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Dropping one hypothesis

- ccc manifold M of dimension $n \geq 3$
- discrete group G acting on M
- consider G-invariant metrics on M

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Dropping one hypothesis

- ccc manifold M of dimension $n \geq 3$
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Open Question: Does $\lambda_1(g)\text{Vol}(g)^{2/n}$ become arbitrarily large?

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An extrinsic constraint

Setup:

- $\bar{M}$, compact smooth manifold of dimension $n \geq 2$
- $X : \bar{M} \to \mathbb{R}^{n+p}$, $p \geq 1$, an immersion
- $M = X(\bar{M})$, compact connected submanifold without boundary immersed in Euclidean space
- g, Riemannian metric naturally induced on $\bar{M}$

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Can view two-dimensional results given earlier in this context

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Can view two-dimensional results given earlier in this context
Result of Colbois-Dodziuk + Nash embedding theorem implies

$$\sup_{M} \lambda_1(M)\text{Vol}(M)^{2/n} = \infty$$

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Spheres appear again

Theorem (Colbois-D-El Soufi)

Let M be a compact convex hypersurface in $\mathbb{R}^{n+1}$. Then

$$\lambda_1(M)\,Vol(M)^{2/n} \leq A(n)\lambda_1(S^n)\,Vol(S^n)^{2/n},$$

where $\lambda_1(S^n) = n$ and $A(n) = \frac{(n+2)\,Vol(S^n)}{2\,Vol(S^{n-1})}$.

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Proof uses barycentric methods and projection

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Replacing "convex"

Hypersurface *M*: intersection index is maximum number of collinear points in *M*

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Replacing "convex"

Hypersurface M: intersection index is maximum number of collinear points in M

Submanifold M^n in $\mathbb{R}^{n+p}$: intersection index of M is

$$i(M) = \sup_{\Pi} \# M \cap \Pi,$$

where Π runs over set of p-planes transverse to M in $\mathbb{R}^{n+p}$

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Using the intersection index

Theorem (Colbois-D-El Soufi)

Let M^n be a compact immersed submanifold of a Euclidean space $\mathbb{R}^{n+p}$. Then

$$\lambda_1(M)\,Vol(M)^{2/n} \le A(n)\left(\frac{i(M)}{2}\right)^{1+\frac{2}{n}}\lambda_1(S^n)\,Vol(S^n)^{2/n}.$$

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Moving on up...

Theorem (Colbois-D-El Soufi)

For every compact n-dimensional immersed submanifold M of $\mathbb{R}^{n+p}$ and for every integer k,

$$\lambda_k(M) Vol(M)^{2/n} \leq c(n) i(M)^{2/n} k^{2/n},$$

where c(n) is an explicit constant depending only on the dimension n.

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What does it all mean?

Combining Colbois-Dodziuk with our results in the extrinsic
context says...

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What does it all mean?

Combining Colbois-Dodziuk with our results in the extrinsic context says...

Given a smooth manifold $\bar{M}$ of dimension $n \geq 3$, there exist Riemannian metrics g of volume 1 on $\bar{M}$ such that any immersion of $\bar{M}$ into a Euclidean space $\mathbb{R}^{n+p}$ which preserves g must have a very large intersection index and volume which concentrates into a small Euclidean ball.

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Summary

- One physical isoperimetric problem is to extremize λ_1 subject to certain constraints, the most basic of which is the volume of the manifold.

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Summary

- One physical isoperimetric problem is to extremize λ_1 subject to certain constraints, the most basic of which is the volume of the manifold.
- The Rayleigh quotient and spheres often play key roles in the solutions to this isoperimetric problem.

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- One physical isoperimetric problem is to extremize λ_1 subject to certain constraints, the most basic of which is the volume of the manifold.
- The Rayleigh quotient and spheres often play key roles in the solutions to this isoperimetric problem.
- For manifolds of dimension at least three, getting bounds on λ_1 requires adding more constraints, either intrinsic (like invariance of the metric and eigenfunctions under a group action) or extrinsic (like immersed submanifolds).

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- Outlook
    - Are there other natural constraints, either of an intrinsic or extrinsic nature, that give interesting results?
    - When upper bounds exist, can we show that they are optimal?

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