Hearing Delzant polygons from the equivariant spectrum

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Outline

Spectral geometry meets symplectic geometry

Abreu: Can one hear the shape of a Delzant polytope?

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More rigorously...

- M^{2n} is toric manifold, i.e., symplectic manifold with "compatible" $\mathbb{T}^n$-action
- g, toric Kähler metric on M
- Delzant/moment polytope associated to M

Abreu: Let M be a toric manifold equipped with a toric Kähler metric g. Does the spectrum of the Laplacian Δ_g determine the moment polytope of M?

An Example

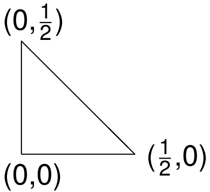
- $M = \mathbb{CP}^2$, equipped with the Fubini-Study form ω_{FS}
- $\mathbb{T}^2$-action on $\mathbb{CP}^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{-i\theta_1} z_1, e^{-i\theta_2} z_2]$$

- moment map

$$\phi[z_0, z_1, z_2] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right)$$

- moment polygon $P = \phi(\mathbb{CP}^2)$, is Delzant



Equivariant version of Abreu's question

Replace Laplace spectrum by equivariant Laplace spectrum

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Fix $\theta \in \mathbb{R}^n$ and $\lambda \in \mathrm{Spec}(M, g)$. Then $\psi(e^{i\theta})$ induces an action on eigenspace corresponding to λ.

Action splits according to weights

Equivariant spectrum = spectrum + weights of actions induced by $\psi(e^{i\theta})$ on eigenspaces for all $\theta \in \mathbb{R}^n$

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Question: Does the equivariant spectrum determine the moment polytope?

The Main Result

Theorem (D–V. Guillemin–R. Sena-Dias)

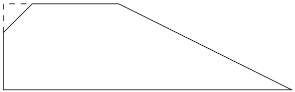
Let M^4 be a toric symplectic manifold with a fixed torus action and a toric metric. Given the equivariant spectrum of M and the spectrum of the associated real manifold, we can reconstruct the moment polygon P of M up to two choices and up to translation for generic polygons with no more than 2 pairs of parallel sides.

There exist such polygons

Theorem (D–V. Guillemin–R. Sena-Dias)

The set of all Delzant polygons in $\mathbb{R}^2$ with $d \geq 5$ sides and at most three pairs of parallel sides is a nonempty, proper open set in the set of all Delzant polygons in $\mathbb{R}^2$.

Note: All Delzant polygons with at least four sides have at least one pair of parallel sides.



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- θ generic: F_θ is pre-image via moment map of vertices of P
- θ (nonzero) multiple of normal to facet of P: F_θ contains union of pre-image of corresponding facet $+$ pre-image of vertices of P
- any θ: F_θ is union of pre-images of all faces to which it is normal with respect to face of lower codimension

Heat invariants

Donnelly (Math Ann., 1976) gave asymptotic expansion of heat trace in presence of isometry

Use this in our setting to show...

Proposition

For each non-generic $\theta \in \mathbb{R}^n$, the equivariant spectrum determines a volume and an unsigned normal vector corresponding to the face(s) of minimal codimension associated to F_θ.

Note: unique face $\Rightarrow$ volume of face; else, get sum of volumes of parallel faces

Spectral data for polygons

Equivariant spectrum $+$ real spectrum give data $\mathcal{D}$:

1. number of edges;
2. (unsigned) normal directions to edges;
3. sums of lengths of edges with each normal direction;
4. volume of polygon

Polygon reconstruction from spectral data

Lemma (Most Obtuse Angle Lemma)

Given a list of vectors $e_1, \ldots, e_d$ in $\mathbb{R}^2$ which are known to be the edges of a convex polygon and u_1 normal to e_1, there is a unique convex polygon P_{e_1,u_1} satisfying

- $0 \in P_{e_1,u_1}$;
- $e_1 \in P_{e_1,u_1}$;
- *u_1 points outward from P_{e_1,u_1};*
- *the ordered list of edges of P_{e_1,u_1} is $e_1, \ldots, e_d$.*

How many possibilities?

Suppose given spectral data $\mathcal{D}$ and allowable edge lists $e_1, \ldots, e_d$ and $-e_1, \ldots, -e_d$. How many possible Delzant polygons are there?

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All are translates of P_{e_1,u_1} or $P_{e_1,-u_1}$.

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Perturb if necessary

Generic: P has unique choice of repeated normals and does not contain subpolygons

Open questions and future directions

- Optimality of results
  - Can the two possibilities be distinguished using spectral data?
  - Are the genericity assumptions necessary?
- Generalization to higher-dimensional polytopes
- What can we say about the metric?
- Inverse problem for generic toric orbifolds