Hearing the geometry of orbifolds

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Outline

Historical Motivation

- Chemistry: identify elements by spectral "fingerprints"
- Physics: development of quantum mechanics
- Mathematics: how are knowledge of structure and knowledge of spectrum related?

A wise man once said...

Sir Arthur Schuster, 1882:

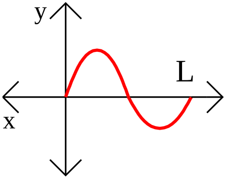
We know a great deal more about the forces which produce the vibrations of sound than about those which produce the vibrations of light. To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction.

String Setup

String of length L with uniform density and tension

Fix endpoints of string

Pluck the string:



Describe motion of string with function $f(x, t)$

Wave equation:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

acceleration curvature

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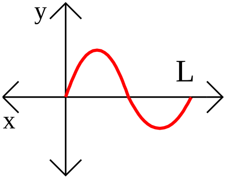

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Solving the Wave Equation

Look for stationary solutions: $f(x,t) = g(x)h(t)$

Substitute such a solution into wave equation $\left(\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}\right)$ to get

$$g(x)h''(t) = g''(x)h(t)$$

or

$$\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)}$$

$$= -\lambda$$

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Rewrite $\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)} = -\lambda$ as

1. $g''(x) = -\lambda g(x)$
2. $h''(t) = -\lambda h(t)$

General solutions are

1. $g(x) = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$
2. $h(t) = C \sin \sqrt{\lambda} t + D \cos \sqrt{\lambda} t$

Boundary conditions imply

1. $g(x) = A \sin \sqrt{\lambda} x, \qquad \sqrt{\lambda} L = n\pi$
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We have $\sqrt{\lambda}L = n\pi$ from boundary conditions on $g(x)$

Frequency of oscillation given by $h(t)$ is $\frac{\sqrt{\lambda}}{2\pi}$

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$$\text{frequency} = \frac{\sqrt{\lambda}}{2\pi} = \frac{n}{2L},$$

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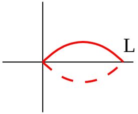
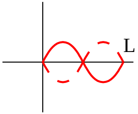
Waveforms

Specific waveforms oscillate at specific frequencies
$\frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \cdots$



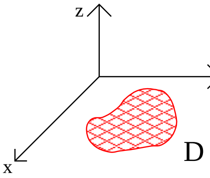


Waveforms form basis for vector space of motion functions $f(x,t)$

Can “hear” the shape (length) of a string!

Drum Setup

D = compact domain in Euclidean plane



- Describe motion with function $f(x, y, t)$
- $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} := \Delta f$
- Sound of drum given by list of frequencies associated to waveforms $f(x, y, t) = g(x, y)h(t)$
- Vibration frequencies = Eigenvalues of Δ on D

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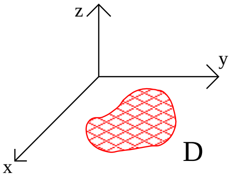

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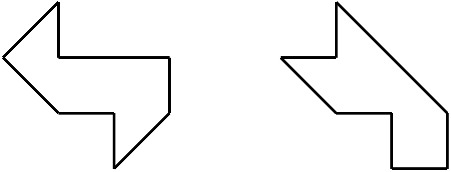
Can one hear the shape of a drum?

- Cannot hear the shape of a drum



D_1 D_2

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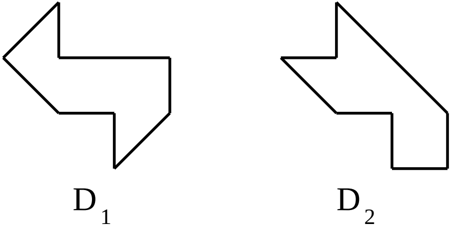

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We begin again. . .

- M is a compact Riemannian manifold
- $\Delta = -div\ grad$
- How much geometric information about M is encoded in the eigenvalue spectrum of Δ?
- Some answers:
  - dimension
  - volume
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What is an orbifold?

- Manifolds
- M/Γ, where Γ is a group acting "nicely" on a manifold M
- $M = S^2$
  Γ is group of rotations of order 3 about north-south axis
  M/Γ is a $(3,3)$-football

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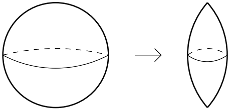
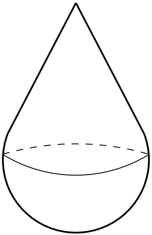


"Bad" Orbifolds

$\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Riemannian Orbifolds

Construction of Riemannian metric on O:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\mathrm{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$
- Each eigenvalue λ_i has finite multiplicity.
- Orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions

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The Big Question

O = compact Riemannian orbifold
$\Delta = -div\ grad$ (locally)

How much topological or geometric information about O is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- orbisurfaces: genus???
- isotropy type???

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Heating Things Up

Heat operator L on O defined by $L = \Delta + \partial/\partial t$

Heat equation: $Lu = 0$

We say that $K : (0, \infty) \times O \times O \to \mathbf{R}$ is a *fundamental solution* of the heat equation, or *heat kernel*, if it satisfies:

1. K is C^0 in the three variables, C^1 in the first, and C^2 in the second;
2. $(\frac{\partial}{\partial t} + \Delta_x)K(t, x, y) = 0$ where Δ_x is the Laplacian with respect to the second variable;
3. $\lim_{t \to 0^+} K(t, x, \cdot) = \delta_x$ for all $x \in O$.

Asymptotic Expansion of Heat Trace

Theorem (D-Gordon-Greenwald-Webb)

Let O be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \dots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to

$$I_0 + \sum_{N \in S(O)} \frac{I_N}{|Ist(N)|} \tag{1}$$

where $S(O)$ is the set of C-strata of O. This asymptotic expansion is of the form

$$(4\pi t)^{-dim(O)/2} \sum_{j=0}^{\infty} c_j t^{\frac{j}{2}}. \tag{2}$$

Huh?!?

I_0 is the "smooth" part, i.e.

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k$$

$a_k(O)$ are the usual heat invariants, e.g.

- $a_0(O) = vol(O)$
- $a_1(O) = \frac{1}{6} \int_O \tau(x) dvol_O(x)$
- If O is finitely covered by a Riemannian manifold M, say $O = G \backslash M$, then

$$a_k(O) = \frac{1}{|G|} a_k(M).$$

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The Singular Part

I_N is the "singular" part:

$$I_N = \sum_{\gamma \in Ist^*(N)} I_{N,\gamma}$$

where

$$I_{N,\gamma} := (4\pi t)^{-dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(\gamma, x) d\, vol_N(x).$$

The b_k's depend on the germ of γ (considered as an isometry of O) and on the Riemannian metric.

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A Simple Application

Theorem (D-Gordon-Greenwald-Webb)

Let O be a Riemannian orbifold with singularities. If O is even-dimensional (respectively, odd-dimensional) and some C-stratum of the singular set is odd-dimensional (respectively, even-dimensional), then O cannot be isospectral to a Riemannian manifold.

Calculating Heat Invariants for 2-Orbifolds

Let O be an orientable two-dimensional orbifold with k cone points of orders $m_1, \cdots, m_k$. Then the first few terms in the asymptotic expansion are:

- degree -1 term:

$$a_0 = vol(O)$$

- degree 0 term:

$$\frac{\chi(O)}{6} + \sum_{i=1}^{k} \frac{m_i^2 - 1}{12 m_i}$$

- degree 1 term:

$$\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{R_{1212}(m_i^4 + 10 m_i^2 - 11)}{360 m_i},$$

where $a_2(O) = \frac{1}{360} \int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$

Teardrops and Footballs

Theorem (D-Gordon-Greenwald-Webb)

Within the class of all footballs (good or bad) and all teardrops, the spectral invariant c is a complete topological invariant. I.e., c determines whether the orbifold is a football or teardrop and determines the orders of the cone points.

Idea of Proof

Define a spectral invariant c as 12 times the degree zero term:

$$c = 2\chi(O) + \sum_{i=1}^{k}(m_i - \frac{1}{m_i})$$

For a teardrop with one cone point of order m, we have

$$c(m) = 2 + m + \frac{1}{m}.$$

For a football with cone points of order r and s, we have

$$c(r, s) = r + s + \frac{1}{r} + \frac{1}{s}.$$

When is the invariant an integer?

We claim that footballs are distinguishable from teardrops.
Suppose $c(m) = c(r,s)$. Then

$$\begin{aligned} m + 2 &= r + s && (3) \\ \frac{1}{m} &= \frac{1}{r} + \frac{1}{s} && (4) \end{aligned}$$

Contradiction!

Claim: $c(r,s)$ determines r and s

- Read off $r+s$ and $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$
- $c(r,s)$ determines $r+s$ and rs
- $(r-s)^2 = (r+s)^2 - 4rs$, so $c(r,s)$ determines $|r-s|$

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Nonnegative Euler Characteristic

Theorem

Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(O) \geq 0$. *The spectral invariant c is a complete topological invariant within C and moreover, it distinguishes the elements of C from smooth oriented closed surfaces.*

Weighted Projective Spaces

Let $\mathbf{N} = (N_1, \ldots, N_{m+1})$ be a vector of positive integers which are pairwise relatively prime. The weighted projective space

$$\mathbb{C}P^m(\mathbf{N}) := \mathbb{C}P^m(N_1, \ldots, N_{m+1}) := (\mathbb{C}^{m+1})^* / \sim,$$

where

$$((z_1, \ldots, z_{m+1}) \sim (\lambda^{N_1} z_1, \ldots, \lambda^{N_{m+1}} z_{m+1}),\ \lambda \in \mathbb{C}^*),$$

is a compact orbifold. It has $m+1$ isolated singularities at the points $[1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]$, with isotropy groups $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_{m+1}}$.

Note that $\mathbb{C}P^m(\mathbf{1})$ is the usual smooth projective space $\mathbb{C}P^m$.

Heat Invariants for Weighted Projective Planes

$O = \mathbb{C}P^2(N_1, N_2, N_3)$ is a weighted projective plane

N_1, N_2, N_3 pairwise relatively prime

Then the first few terms in the asymptotic expansion are:

- degree -2 term: $a_0 = vol(O)$
- degree -1 term: $a_1 = \frac{1}{6}\int_O \tau dvol_O(g)$
- degree 0 term: $\frac{a_2}{16\pi^2} + b_0$, where

$$a_2(O) = \frac{1}{360}\int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$$

and b_0 involves N_1, N_2, N_3.

Listening to Weighted Projective Planes

Theorem (Abreu-D-Freitas-Godinho)

Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities, equipped with any Kähler orbifold metric. Then the spectra of its Laplacian acting on functions and 1-forms determine the weights N_1, N_2 and N_3.

Tools in Proof:

- Heat invariants
- Localization in equivariant cohomology
- Expression for Kähler metrics
- Elementary number theory

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Summary

- Big Question: How much topological or geometric information about an object is encoded in the eigenvalue spectrum of Δ?
- We have an asymptotic expansion of the heat trace for orbifolds.
- The heat invariants can be combined with other tools to tell us that certain classes of orbifolds contain objects that are spectrally distinguished.
- Outlook
  - Other classes of orbifolds to which this strategy could be successfully applied?
  - Examples of isospectral orbifolds with “interesting” features