Orbifolds and isometries

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Workshop on Global Riemannian Geometry, Orbifolds, and Related Topics
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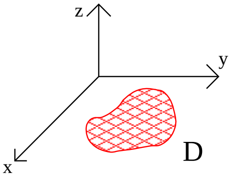
Joint work with Victor Guillemin and Rosa Sena-Dias

Outline

Vibrating Drumheads

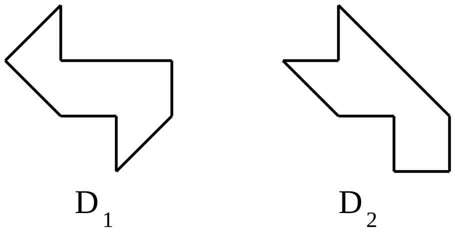
D = compact domain in Euclidean plane



- Vibration frequencies $\leftrightarrow$ Eigenvalues of Δ on D
- How much geometry is encoded in the spectrum?

Can one hear the shape of a drum?

- Cannot hear the shape of a drum



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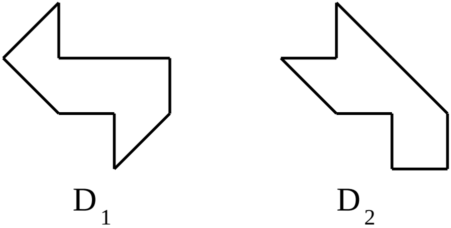

- Can hear area and perimeter of drumhead
- Orbifolds arise in construction

Listening to orbifolds

O = compact Riemannian orbifold
$\Delta = -div\ grad$ (locally)

How much topological or geometric information about O is encoded in the eigenvalue spectrum of Δ?

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Answers:

- dimension
- volume
- isotropy type: work by Stanhope, Shams-Stanhope-Webb, Rossetti-Schueth-Weilandt

Distinguishing Orbifolds

Define a spectral invariant c as 12 times the degree zero term in the asymptotic expansion of the heat trace:

$$c = 2\chi(O) + \sum_{i=1}^{k}(m_i - \frac{1}{m_i})$$

Theorem (D-Gordon-Greenwald-Webb)

Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(O) \geq 0$. The spectral invariant c is a complete topological invariant within C and moreover, it distinguishes the elements of C from smooth oriented closed surfaces.

Spectral geometry + Symplectic geometry

Abreu: Can one hear the shape of a Delzant polytope?

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- M, toric manifold
- g, toric Kähler metric on M
- Delzant polytope associated to M

An Example

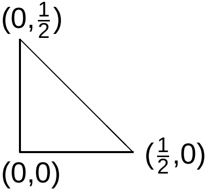
- $M = \mathbb{CP}^2$, equipped with the Fubini-Study form ω_{FS}
- $\mathbb{T}^2$-action on $\mathbb{CP}^2$ given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{-i\theta_1} z_1, e^{-i\theta_2} z_2]$$

- moment map

$$\phi[z_0, z_1, z_2] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right)$$

- moment polygon $P = \phi(\mathbb{CP}^2)$, is Delzant



An orbifold example

- $M = \mathbb{CP}^m(\mathbf{N})$, where $\mathbf{N} = (N_1, \cdots, N_{m+1})$
- $\mathbb{T}^n$-action on M^{2n} may have fixed points
- moment map
- moment polytope is not necessarily Delzant

Abreu: Can one hear the moment polytope and its labels?

Weighted Projective Spaces

Theorem (Abreu-D-Freitas-Godinho)

Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities, equipped with any Kähler orbifold metric. Then the spectra of its Laplacian acting on functions and 1-forms determine the weights N_1, N_2 and N_3.

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Theorem (Guillemin-Uribe-Wang)

In many cases, one can “hear” the weights of a weighted projective space.

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Replace Laplace spectrum by equivariant Laplace spectrum

Equivariant spectrum = spectrum + weights of representation of $\mathbb{T}^n$ on eigenspaces

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Donnelly's theorem + heat trace for orbifolds = equivariant heat trace for orbifolds

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Theorem (D-Guillemin-Sena-Dias)

In the symplectic toric setting, under a genericity assumption, the spectrum determines the normals to and volumes of the facets of the polytope, as well as the labels on the polytope.

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Minkowski problem

Extending these ideas

Goal: equivariant version of orbifold heat trace for general group actions, i.e., not necessarily abelian

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Issue: How do the orbifold group actions and isometry interact?

Can isotropy type vary within a single stratum of the fixed point set of an isometry (other than at endpoints)?