Can you hear the metric on a sphere?

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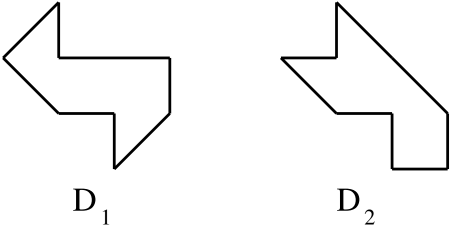
AMS Special Session on Geometry of Manifolds, Singular Spaces, and Groups
Michigan State University
15 March 2015

Can one hear the shape of a drum?

Carolyn Gordon, David Webb and Scott Wolpert (1992):



Question (Miguel Abreu, c. 2000): Can one hear the shape of a Delzant polytope?

Spectral geometry meets symplectic geometry

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Theorem (Delzant)

The moment/Delzant polytope of a symplectic toric manifold M determines M up to symplectomorphism.

Question (Abreu, more technically): Let M^{2n} be a symplectic toric manifold equipped with a $\mathbb{T}^n$-invariant Riemannian metric g. Does Spec(Δ_g) determine the moment polytope of M?

Metric questions in spectral geometry

- Victor Guillemin and David Kazhdan ('80): compact negatively curved Riemannian surface is spectrally rigid
- Christopher Croke and Vladimir Sharafutdinov ('98): compact negatively curved Riemannian manifold is spectrally rigid

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- Jochen Brüning and Ernst Heintze ('84): metrics on S^2 invariant under S^1 and under reflection in xy-plane; TFAE: isometric; isospectral; same S^1-invariant spectrum
- Steve Zelditch ('98): recovers rotationally symmetric metric on S^2 from the spectrum under certain hypotheses; uses wave trace techniques
- D–Guillemin–Sena-Dias ('12): on generic toric orbifold with toric Kähler metric, equivariant spectrum "hears" if metric has constant scalar curvature

Focus on S^2 with S^1-invariant metrics

Set of S^1-invariant metrics on S^2 can be parametrized by *symplectic potential* g

Given $g : (-1, 1) \to \mathbb{R}$ and action-angle coordinates (x, θ), metric written as

$$\ddot{g} dx \otimes dx + \frac{d\theta \otimes d\theta}{\ddot{g}}$$

Spectral determination

Theorem (D–Macedo–Sena-Dias)

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Asymptotic means that we only need to consider large weights

The magic bullet

- $\lambda_i(\alpha, h)$ are eigenvalues of $h^2\Delta$ restricted to

$$C^\infty(M)^{\frac{\alpha}{h}} = \left\{ f \in C^\infty(M) : f(gp) = \chi_{\frac{\alpha}{h}}(g) f(p) \right\}$$

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- Spectral measure $\mu_{\frac{\alpha}{h}}(\rho) = \sum_i \rho(\lambda_i(\alpha, h))$, $\rho \in C_0^\infty(\mathbb{R})$
- Asymptotic expansion of this measure, expressed in terms of a Schwartz kernel, is determined by the asymptotic equivariant spectrum

A little history

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- Yves Colin de Verdière: spectrum of Schrödinger operator on $\mathbb{R}^2$ with single well potential determines the potential
- Victor Guillemin and Zuoqin Wang: generalized Colin de Verdière's result to double wells

It's harder than it sounds: first spectral invariant

Let ρ be a compactly supported smooth function on $\mathbb{R}$. Let $v = \ddot{g}$ and $\tau = \frac{\xi^2}{v} + \alpha^2 v$. Then

$$\int_{[-1,1]\times\mathbb{R}} \rho(\tau)\,dx\,d\xi$$

is asymptotically spectrally determined.

It's harder than it sounds: second spectral invariant

The expression

$$\frac{1}{2}\int_{[-1,1]\times\mathbb{R}} \frac{1}{v}\left[\xi^2\left(\frac{v^{(2)}}{v^2}-\frac{2(v')^2}{v^3}\right)-\alpha^2 v^{(2)}\right]\rho^{(2)}(\tau)\mathrm{d}\xi\mathrm{d}x$$
$$-\frac{2}{3}\int_{[-1,1]\times\mathbb{R}} \frac{\xi^2}{v}\left[\xi^2\left(\frac{3(v')^2}{v^4}-\frac{v^{(2)}}{v^3}\right)+\alpha^2\left(\frac{v^{(2)}}{v}-\frac{(v')^2}{v^2}\right)\right]\rho^{(3)}(\tau)\mathrm{d}\xi\mathrm{d}x$$
$$-\frac{1}{3}\int_{[-1,1]\times\mathbb{R}} \frac{(v')^2}{v}\left(-\frac{\xi^2}{v^2}+\alpha^2\right)^2\rho^{(3)}(\tau)\mathrm{d}\xi\mathrm{d}x$$
$$-\frac{1}{2}\int_{[-1,1]\times\mathbb{R}} \frac{\xi^2(v')^2}{v^2}\left(-\frac{\xi^2}{v^2}+\alpha^2\right)^2\rho^{(4)}(\tau)\mathrm{d}\xi\mathrm{d}x$$

is also asymptotically spectrally determined.

Exploiting these two quantities

Key tools:

- Abel transform using first invariant to get minimum of $\ddot{g}$ at $x = 0$ aespectrally determined
- Integration by parts with respect to ξ in second invariant
- Show an initial value ODE has at most one solution

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- Luck

Future directions

- Would more spectral invariants help us recover metrics corresponding to finitely many wells?
- What do these invariants reveal about higher-dimensional toric manifolds or orbifolds?
- Abreu's original question: Can one hear the shape of a Delzant polytope?

Thanks!