Eigenvalue (mis)behavior on manifolds

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May 7, 2009

Outline

1. Isoperimetric inequalities
2. Upper bounds on eigenvalues for manifolds
3. Metrics invariant under a group action
4. Submanifolds

The Original Isoperimetric Inequality

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  - planar
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  - fix length L, maximize area A
  - "The" isoperimetric inequality:

$$L^2 \geq 4\pi A$$

Generalizations

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- Mathematical physics: a physical quantity is extremal for a circular or spherical domain

An example

Setup:

- domain $D \subset \mathbb{R}^2$
- $f : D \to \mathbb{R}$, a smooth function which equals zero on the boundary of D
- $\Delta f := \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$

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Seek solutions to $\Delta f = \lambda f$

Especially interested in λ_1

The Rayleigh quotient for domains

Theorem

Let D be a domain with Δ acting on piecewise smooth, nonzero functions f which are zero on the boundary of D, and with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$. For any such f,

$$\lambda_1 \leq \frac{\int_D |\nabla f|^2}{\int_D f^2},$$

with equality if and only if f is an eigenfunction of λ_1.

Minima of the Rayleigh quotient

Theorem (Rayleigh, Faber-Krahn)

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Higher-dimensional analog: Rayleigh quotient attains minimum iff $D \subset \mathbb{R}^n$ is sphere

The Rayleigh quotient for manifolds

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- (M, g), compact Riemannian manifold
- Δ, Laplace operator on (M, g)
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- Rayleigh quotient:

$$\lambda_1(M) = \inf_{f \in \mathcal{F}_1} \frac{\int_M |\nabla f|^2}{\int_M f^2},$$

where $\mathcal{F}_1$ is set of smooth nonzero functions on M orthogonal to the constant functions

Hersch's Theorem

Theorem (Hersch)

Consider the sphere S^2 equipped with any Riemannian metric g. We have

$$\lambda_1 Vol(g) \leq 8\pi,$$

with equality only in the case of the constant curvature metric.

Idea of proof: Move S^2 to its center of mass, and use coordinate functions as test functions in the Rayleigh quotient.

Compact orientable surfaces

Theorem (Yang-Yau)

Let (M, g) be a compact orientable surface of genus γ. Then

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Generalized to nonorientable surfaces by Li-Yau

What's changed?

Theorem (Korevaar)

Let (M, g) be a compact orientable surface of genus γ, and let $C > 0$ be a universal constant. For every integer $k \geq 1$,

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Open questions abound, e.g., *optimal* bound for λ_2 on Klein bottle or surface of genus 2

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Theorem (Colbois-Dodziuk)

Let (M^n, g) be a compact, closed, connected manifold of dimension at least three. Then

$$\sup \lambda_1(g) Vol(g)^{2/n} = \infty,$$

where the supremum is taken over all Riemannian metrics g on M.

Idea of proof

- Use Bleecker's result: take (S^n, g_0) such that $\text{Vol}(S^n, g_0) = 1$ and $\lambda_1(g_0) \geq k + 1$, where k is a large constant

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- Take arbitrary metric g_1 on M whose restriction to Ω equals g_0 restricted to Ω, make g_1 really small on most of $M \setminus \Omega$ without changing it on Ω
- M "looks like" (S^n, g_0), and λ_1 for modified g_1 is like $\lambda_1(g_0)$

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- intrinsic constraints: restrict to conformal class of metrics, to projective Kähler metrics, to metrics which preserve the symplectic or Kähler structure, etc.
- extrinsic constraints: mean curvature (Reilly's inequality)

Back to the 2-sphere

Tweak Hersch's assumptions:

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Bound is attained by the union of two disks of equal area

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Theorem (Colbois-D-El Soufi)

Let (S^n, g) be as above, with $Vol(g) = 1$. Then, for all $k \in \mathbb{Z}$,

$$\lambda_k^{O(n)}(g) < \lambda_k^{O(n)}(D^n)\, Vol(D^n)^{2/n},$$

where D^n is the Euclidean n-ball of volume $1/2$.

What about *any* manifold, not just spheres?

- replace S^n by ccc manifold M of dimension $n \geq 3$
- replace $O(n)$ by finite subgroup G of group of diffeomorphisms acting on M
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Then $\lambda_1^G(g)\mathrm{Vol}(g)^{2/n}$ is unbounded!

Proof: apply Colbois-Dodziuk "equivariantly"

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Open Question: Does $\lambda_1(g)\mathrm{Vol}(g)^{2/n}$ become arbitrarily large?

An extrinsic constraint

Hypersurfaces: curve in plane, two-dimensional surface in $\mathbb{R}^3$

Submanifolds: equator in S^2, manifold in $\mathbb{R}^k$ for k sufficiently large

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Why *extrinsic*?

Spheres appear again

Theorem (Colbois-D-El Soufi)

Let M be a compact convex hypersurface in $\mathbb{R}^{n+1}$. Then

$$\lambda_1(M)\,Vol(M)^{2/n} \leq A(n)\lambda_1(S^n)\,Vol(S^n)^{2/n},$$

where $\lambda_1(S^n) = n$ and $A(n) = \frac{(n+2)\,Vol(S^n)}{2\,Vol(S^{n-1})}$.

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Proof uses barycentric methods and projection

Replacing “convex”

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Submanifold M^n in $\mathbb{R}^{n+p}$: intersection index of M is

$$i(M) = \sup_{\Pi} \# M \cap \Pi,$$

where Π runs over set of p-planes transverse to M in $\mathbb{R}^{n+p}$

Using the intersection index

Theorem (Colbois-D-El Soufi)

Let M^n be a compact immersed submanifold of a Euclidean space $\mathbb{R}^{n+p}$. Then

$$\lambda_1(M)\,Vol(M)^{2/n} \leq A(n)\left(\frac{i(M)}{2}\right)^{1+\frac{2}{n}} \lambda_1(S^n)\,Vol(S^n)^{2/n}.$$

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- One physical isoperimetric problem is to extremize λ_1 subject to certain constraints, the most basic of which is the volume of the manifold.
- The Rayleigh quotient and spheres often play key roles in the solutions to this isoperimetric problem.
- For manifolds of dimension at least three, getting bounds on λ_1 requires adding more constraints, either intrinsic (like invariance of the metric and eigenfunctions under a group action) or extrinsic (like immersed submanifolds).
- Outlook
  - Are there other natural constraints, either of an intrinsic or extrinsic nature, that give interesting results?
  - When upper bounds exist, can we show that they are optimal?

References

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