Are right spherical triangles wrong?

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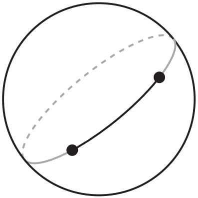
Basic objects

- Unit sphere: set of points that are 1 unit from the origin in $\mathbb{R}^3$
- Straight lines $\rightarrow$ great circles
- Line segments $\rightarrow$ great circle arcs
  - Any pair of points on the sphere has at least two line segments connecting the points.
  - Can you find a pair of points that has more than two line segments connecting the points?



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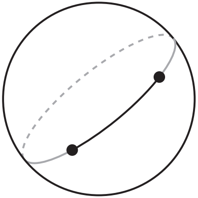
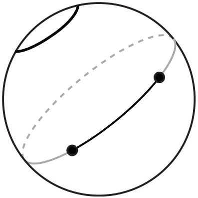


More basic objects

- Angles: measure angle in tangent plane between tangents to line segments
- Circle: intersection of sphere with any plane that's not tangent to sphere; have great and small



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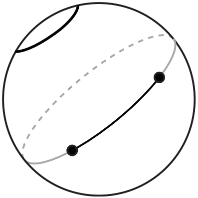
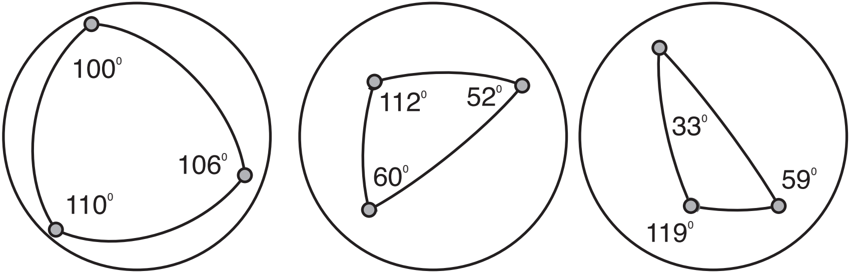


Spherical triangles

A spherical triangle is the union of three line segments joining three non-collinear points.



Properties of spherical triangles:

- Sum of the angles is not π!
- Girard's Theorem: Area(T) = sum of angles $- \pi$
- Similarities with Euclidean triangles: Side-Side-Side, Side-Angle-Side, Isosceles Triangle Theorems still hold

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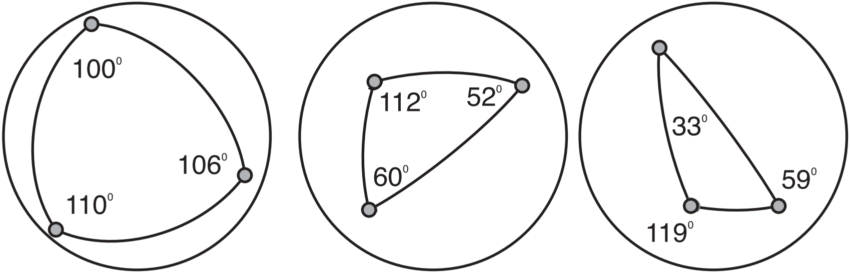

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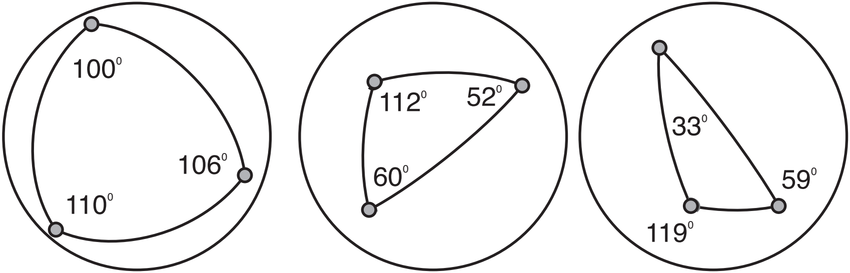

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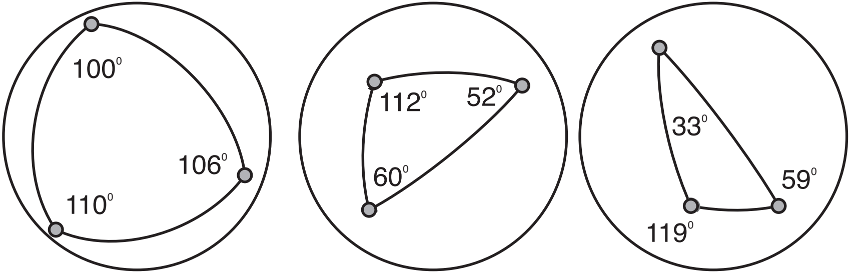

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Right triangles

- Euclidean right triangles: one angle has measure $\pi/2$
- Traditional spherical right triangles: at least one angle has measure $\pi/2$
- Dissimilarities between Euclidean and traditional spherical right triangles:
  - Angles inscribed in semicircles vary in size
  - Spherical Pythagorean Theorem: Let ABC be a traditional spherical right triangle with $\angle C = \frac{\pi}{2}$. Then

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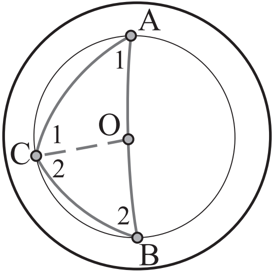
A new definition



Half-sum triangle: triangle in which one angle is the sum of the other two

Is a half-sum triangle the *right* right triangle on the sphere?

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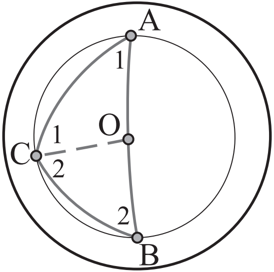

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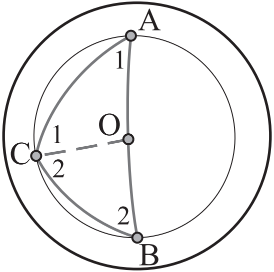

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Angles inscribed in semicircles

- Let ABC be a spherical half-sum triangle inscribed in a semicircle.
- How can we describe $\angle C$?

$$
\begin{aligned}
\angle A + \angle B + \angle C - \pi &= \text{Area}(ABC) \\
\angle A + \angle B + (\angle A + \angle B) &= \pi + \text{Area}(ABC) \\
\angle C &= \frac{\pi}{2} + \frac{1}{2}\text{Area}(ABC)
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- Gaussian curvature: $K = 1$ on unit sphere, $K = 0$ in Euclidean plane

$$\angle C = \frac{\pi}{2} + K\frac{\text{Area}(ABC)}{2}$$

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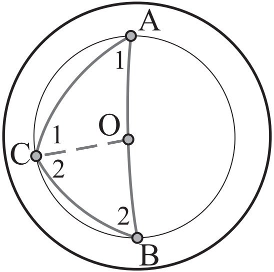
Circumcenter Lemma

Lemma (Dickinson-Salmassi)

Let ABC be a spherical triangle with circumcenter O. Then O is on $\overleftrightarrow{AB}$ *if and only if* $\angle A + \angle B = \angle C$.



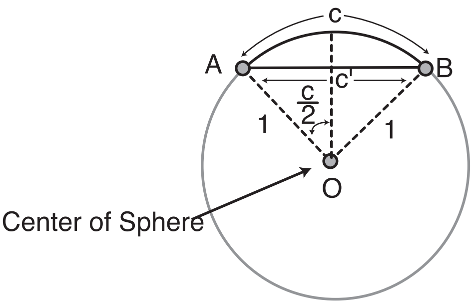
Pythagorean Theorem

Theorem (Dickinson-Salmassi)

In a spherical half-sum triangle ABC with $\angle C = \angle A + \angle B$,

$$\sin^2\left(\frac{a}{2}\right) + \sin^2\left(\frac{b}{2}\right) = \sin^2\left(\frac{c}{2}\right).$$

Claim: $c' = 2\sin(\frac{c}{2})$



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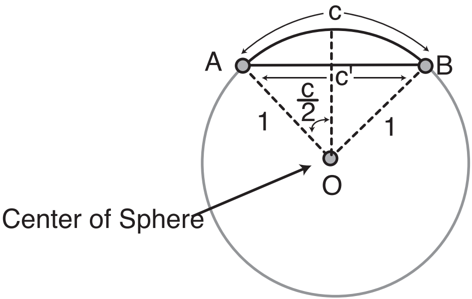

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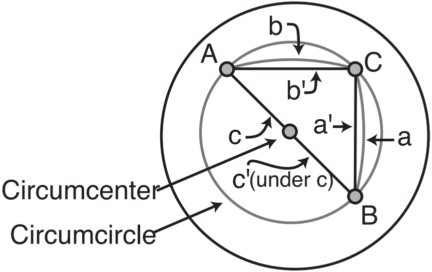
Proving that $a'^2 + b'^2 = c'^2$



Comparing the two Spherical Pythagorean Theorems

Traditional Spherical Pythagorean Theorem: Let ABC be a traditional spherical right triangle with $\angle C = \frac{\pi}{2}$. Then

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Spherical obtuse-sum with $\angle C > \angle A + \angle B$:

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Euclidean obtuse triangle with obtuse angle at C:

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Spherical rectangles

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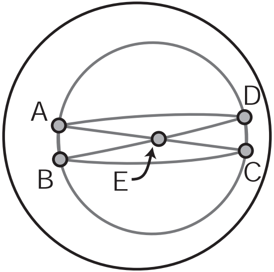
- Spherical quadrilateral with four right angles?
- Rectangle: quadrilateral with four congruent angles



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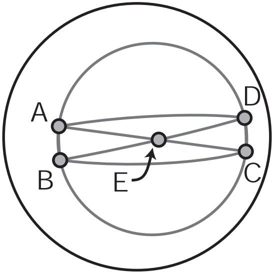

Summary

Dissimilarities (angles inscribed in semicircles vary in size, Spherical Pythagorean Theorem missing squares, spherical rectangles don't split into two right triangles) between Euclidean and spherical right triangles vanish if we make a new definition of a spherical right triangle.

Further Reading:

- Google "geometry sphere"
- William Dickinson and Mohammad Salmassi, The *right* right triangle on the sphere. *College Math. J.* 39 (2008), no. 1, 24-33.