Building polygons from spectral data

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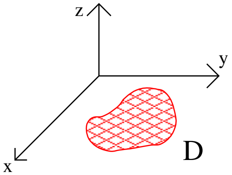
Joint work with Victor Guillemin and Rosa Sena-Dias

Outline

Vibrating Drumheads

D = compact domain in Euclidean plane



- Vibration frequencies $\leftrightarrow$ Eigenvalues of Δ on D
- How much geometry is encoded in the spectrum?

Listening to Polytopes

Abreu: Can one hear the shape of a Delzant polytope?

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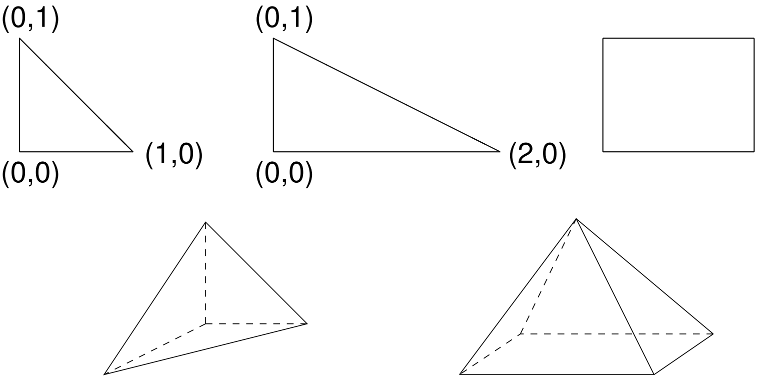
A convex polytope P in $\mathbb{R}^n$ is *Delzant* if

- it is *simple*, i.e., there are n facets meeting at each vertex;
- it is *rational*, i.e., for every facet of P, a primitive outward normal can be chosen in $\mathbb{Z}^n$;
- it is *smooth*, i.e., for every vertex of P, the outward normals corresponding to the facets meeting at that vertex form a basis for $\mathbb{Z}^n$.

Examples and Non-examples



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- Delzant/moment polytope associated to M

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Abreu's question: Let M be a toric manifold equipped with a toric Kähler metric g. Does the spectrum of the Laplacian Δ_g determine the moment polytope of M?

Modifying Abreu's Question: Step 1

A convex polytope P in $\mathbb{R}^n$ is *rational simple* if it is simple, it is rational, and for every vertex of P, the outward normals corresponding to the facets meeting at that vertex form a basis for $\mathbb{Q}^n$.

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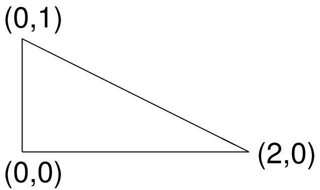

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$$\begin{aligned}\text{Equivariant spectrum} \quad &= \quad \text{Laplace spectrum} \\ &\quad\;\; + \text{weights for each eigenvalue}\end{aligned}$$

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Question

Let M be a toric orbifold equipped with a toric Kähler metric g. Does the equivariant spectrum of the Laplacian Δ_g determine the labeled moment polytope of M?

What We Hear

The equivariant spectrum associated to a toric orbifold M whose moment polytope has no parallel facets determines:

1. the (unsigned) normal directions to the facets;
2. the volumes of the corresponding facets;
3. the labels of the facets.

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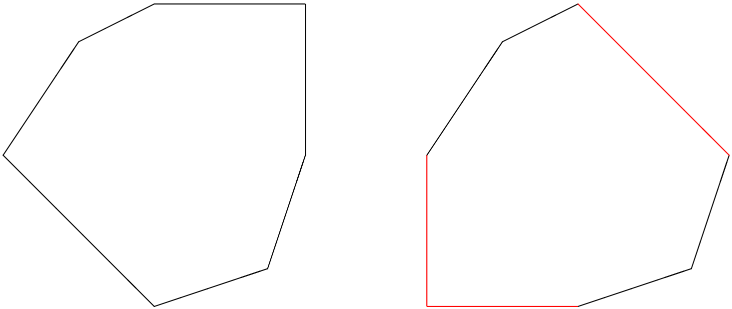
Given this data, how many labeled moment polytopes can you build?

Building Polygons



Minkowski's Theorem

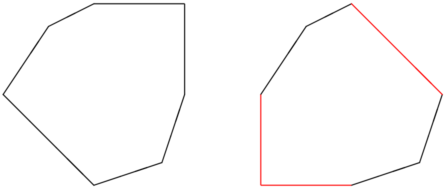
Theorem

(Minkowski; Klain) Given a list $\{(u_i, \nu_i), u_i \in \mathbb{R}^n, \nu_i \in \mathbb{R}^+, \quad i = 1, \ldots, d\}$ where the u_i are unit vectors that span $\mathbb{R}^n$, there exists a convex polytope P with facet normals $u_1, \ldots, u_d$ and corresponding facet volumes $\nu_1, \ldots, \nu_d$ if and only if

$$\sum_{i=1}^{d} \nu_i u_i = 0.$$

Moreover, this polytope is unique up to translation.

Troublemaker 1: subpolytopes



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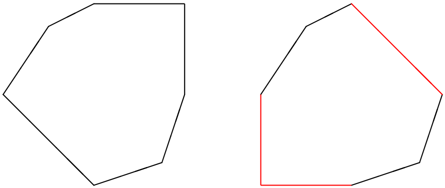

Lemma

Let P be a convex polytope in $\mathbb{R}^n$ with no subpolytopes and facet volumes $\nu_1, \ldots, \nu_d$. Assume that the facet normals to P are $u_1, \ldots, u_d$ up to sign. Then, up to translation, there are only 2 choices for the set of signed normals.

Troublemaker 2: parallel facets

Parallel facets introduce indeterminants:

- know *sum* of volumes of facets in parallel pair
- do not know which normal directions in list are repeated

Bye-bye, troublemaking polytopes

Lemma

Close to any rational simple polytope in $\mathbb{R}^n$, there is a rational simple polytope that has no parallel facets and has no subpolytopes.

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Orbifolds are better than manifolds!

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Theorem

(D–V. Guillemin–R. Sena-Dias) Let M be a generic toric orbifold with a fixed torus action and a toric Kähler metric. Then the equivariant spectrum of M determines the labeled moment polytope P of M, up to two choices and up to translation.

Abreu's original question

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Theorem (D–V. Guillemin–R. Sena-Dias)

Let M^4 be a toric symplectic manifold with a fixed torus action and a toric metric. Given the equivariant spectrum of M and the spectrum of the associated real manifold, we can reconstruct the moment polygon P of M up to two choices and up to translation for generic polygons with no more than 2 pairs of parallel sides.

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