Listening to orbifolds: What does the Laplace spectrum tell us?

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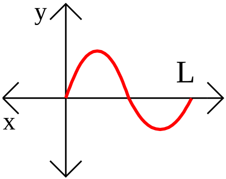
Outline

Historical Motivation

- Chemistry: identify elements by spectral “fingerprints”
- Physics: development of quantum mechanics
- Mathematics: how are knowledge of structure and knowledge of spectrum related?

String Setup

String of length L with fixed endpoints
Pluck the string:



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Wave equation:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

acceleration curvature

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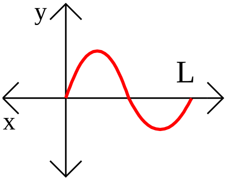

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Waveforms

- Look for waveforms, i.e., solutions $f(x,t)$ such that $f(x,t) = g(x)h(t)$
- Specific waveforms oscillate at specific frequencies $\frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \ldots$
- Waveforms form basis for vector space of motion functions $f(x,t)$
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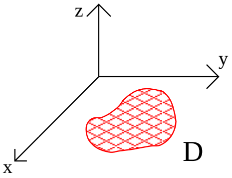
Drum Setup

D = compact domain in Euclidean plane



- Describe motion with function $f(x, y, t)$
- $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} := \Delta f$
- Sound of drum given by list of frequencies associated to waveforms $f(x, y, t) = g(x, y)h(t)$
- Vibration frequencies = Eigenvalues of Δ on D

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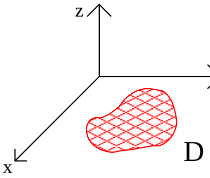

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Can one hear the shape of a drum?

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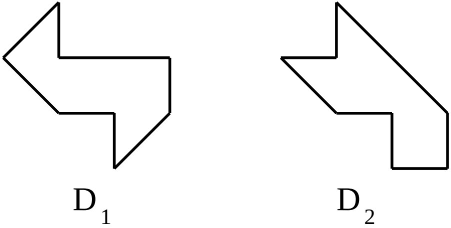

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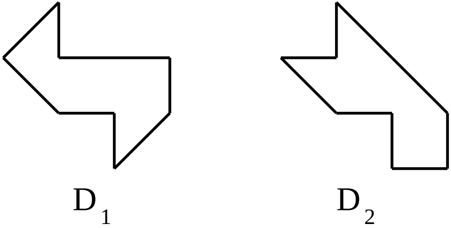

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We begin again. . .

- M is a compact Riemannian manifold
- $\Delta = -div\ grad$
- How much geometric information about M is encoded in the eigenvalue spectrum of Δ?
- Answers:
  - dimension
  - volume
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What is an orbifold?

- Manifolds
- M/Γ, where Γ is a group acting “nicely” on a manifold M
- $M = S^2$
  Γ is group of rotations of order 3 about north-south axis
  M/Γ is a $(3,3)$-football

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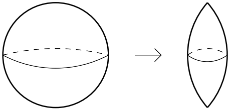
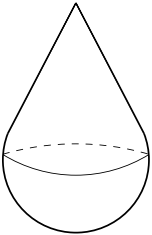


"Bad" Orbifolds

$\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order p



Riemannian Orbifolds

Construction of Riemannian metric on O:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\mathrm{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$
- Each eigenvalue λ_i has finite multiplicity.
- Orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions

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The Big Question

O = compact Riemannian orbifold
$\Delta = -\mathit{div}\ \mathit{grad}$ (locally)

How much topological or geometric information about O is encoded in the eigenvalue spectrum of Δ?

Answers:

- dimension
- volume
- orbisurfaces: genus???
- isotropy type???

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Asymptotic Expansion of Heat Trace

Theorem (D-Gordon-Greenwald-Webb)

Let O be a Riemannian orbifold and let $\lambda_1 \leq \lambda_2 \leq \ldots$ be the spectrum of the associated Laplacian acting on smooth functions on O. The heat trace $\sum_{j=1}^{\infty} e^{-\lambda_j t}$ of O is asymptotic as $t \to 0^+$ to

$$I_0 + \sum_{N \in S(O)} \frac{I_N}{|Ist(N)|} \tag{1}$$

where $S(O)$ is the set of C-strata of O. This asymptotic expansion is of the form

$$(4\pi t)^{-dim(O)/2} \sum_{j=0}^{\infty} c_j t^{\frac{j}{2}}. \tag{2}$$

Huh?!?

I_0 is the “smooth” part, i.e.

$$I_0 = (4\pi t)^{-dim(O)/2} \sum_{k=0}^{\infty} a_k(O) t^k$$

$a_k(O)$ are the usual heat invariants, e.g.

- $a_0(O) = vol(O)$
- $a_1(O) = \frac{1}{6} \int_O \tau(x) dvol_O(x)$
- If O is finitely covered by a Riemannian manifold M, say $O = G \backslash M$, then

$$a_k(O) = \frac{1}{|G|} a_k(M).$$

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The Singular Part

I_N is the "singular" part:

$$I_N = \sum_{\gamma \in Ist^*(N)} I_{N,\gamma}$$

where

$$I_{N,\gamma} := (4\pi t)^{-dim(N)/2} \sum_{k=0}^{\infty} t^k \int_N b_k(\gamma, x) d\, vol_N(x).$$

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A Simple Application

Theorem (D-Gordon-Greenwald-Webb)

Let O be a Riemannian orbifold with singularities. If O is even-dimensional (respectively, odd-dimensional) and some C-stratum of the singular set is odd-dimensional (respectively, even-dimensional), then O cannot be isospectral to a Riemannian manifold.

Calculating Heat Invariants for 2-Orbifolds

Let O be an orientable two-dimensional orbifold with k cone points of orders $m_1, \cdots, m_k$. Then the first few terms in the asymptotic expansion are:

- degree -1 term:

$$a_0 = vol(O)$$

- degree 0 term:

$$\frac{\chi(O)}{6} + \sum_{i=1}^{k} \frac{m_i^2 - 1}{12 m_i}$$

- degree 1 term:

$$\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{R_{1212}(m_i^4 + 10 m_i^2 - 11)}{360 m_i},$$

where $a_2(O) = \frac{1}{360} \int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$

Teardrops and Footballs

Theorem (D-Gordon-Greenwald-Webb)

Within the class of all footballs (good or bad) and all teardrops, the spectral invariant c is a complete topological invariant. I.e., c determines whether the orbifold is a football or teardrop and determines the orders of the cone points.

Idea of Proof

Define a spectral invariant c as 12 times the degree zero term:

$$c = 2\chi(O) + \sum_{i=1}^{k}(m_i - \frac{1}{m_i})$$

For a teardrop with one cone point of order m, we have

$$c(m) = 2 + m + \frac{1}{m}.$$

For a football with cone points of order r and s, we have

$$c(r,s) = r + s + \frac{1}{r} + \frac{1}{s}.$$

When is the invariant an integer?

Suppose $c(m) = c(r, s)$. Then

$$m + 2 = r + s \tag{3}$$
$$\frac{1}{m} = \frac{1}{r} + \frac{1}{s} \tag{4}$$

Contradiction!

Claim: $c(r, s)$ determines r and s

- Read off $r + s$ and $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$
- $c(r, s)$ determines $r + s$ and rs
- $(r - s)^2 = (r + s)^2 - 4rs$, so $c(r, s)$ determines $|r - s|$

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Nonnegative Euler Characteristic

Theorem

Let C be the class consisting of all closed orientable 2-orbifolds with $\chi(O) \geq 0$*. The spectral invariant c is a complete topological invariant within C and moreover, it distinguishes the elements of C from smooth oriented closed surfaces.*

Weighted Projective Spaces

Let $\mathbf{N} = (N_1, \ldots, N_{m+1})$ be a vector of positive integers which are pairwise relatively prime. The weighted projective space

$$\mathbb{C}P^m(\mathbf{N}) := \mathbb{C}P^m(N_1, \ldots, N_{m+1}) := (\mathbb{C}^{m+1})^* / \sim,$$

where

$$((z_1, \ldots, z_{m+1}) \sim (\lambda^{N_1} z_1, \ldots, \lambda^{N_{m+1}} z_{m+1}), \lambda \in \mathbb{C}^*),$$

is a compact orbifold. It has $m+1$ isolated singularities at the points $[1 : 0 : \cdots : 0]$, $\ldots$, $[0 : \cdots : 0 : 1]$, with isotropy groups $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_{m+1}}$.

Note that $\mathbb{C}P^m(\mathbf{1})$ is the usual smooth projective space $\mathbb{C}P^m$.

Heat Invariants for Weighted Projective Planes

$O = \mathbb{C}P^2(N_1, N_2, N_3)$ is a weighted projective plane

N_1, N_2, N_3 pairwise relatively prime

Then the first few terms in the asymptotic expansion are:

- degree -2 term: $a_0 = vol(O)$
- degree -1 term: $a_1 = \frac{1}{6}\int_O \tau dvol_O(g)$
- degree 0 term: $\frac{a_2}{16\pi^2} + b_0$, where

$$a_2(O) = \frac{1}{360}\int_O (2|R|^2 - 2|\rho|^2 + 5\tau^2) dvol_O(g)$$

and b_0 involves N_1, N_2, N_3.

Listening to Weighted Projective Planes

Theorem (Abreu-D-Freitas-Godinho)

Let $M := \mathbb{C}P^2(N_1, N_2, N_3)$ be a four-dimensional weighted projective space with isolated singularities, equipped with any Kähler orbifold metric. Then the spectra of its Laplacian acting on functions and 1-forms determine the weights N_1, N_2 and N_3.

Tools in Proof

- Heat invariants
- Localization in equivariant cohomology
- Expression for Kähler metrics
- Elementary number theory

Summary

- Big Question: How much topological or geometric information about an object is encoded in the eigenvalue spectrum of Δ?
- We have an asymptotic expansion of the heat trace for orbifolds.
- The heat invariants can be combined with other tools to tell us that certain classes of orbifolds contain objects that are spectrally distinguished.

- Outlook
  - Other classes of orbifolds to which this strategy could be successfully applied?
  - Examples of isospectral orbifolds with "interesting" features